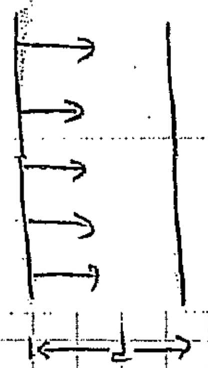
Problem # 1
20 points

Final Exam
NE290 H Spring 2009
Barnard and Lund
Due Monday, May 18th 5:00pm, Evans Room 61

(1) CALCULATE THE CURRENT DENSITY J FOR AN EXTREME RELATIVISTIC 1-D DIODE. (i.e. ignore the small region of the diode for which $\beta \simeq 1$ is NOT A GOOD APPROXIMATION.) Assume space-charge limited emission. LET THE DIODE HAVE LENGTH d, AND voltage V, AND THE ION SPECIES HAVE MASS m AND CHARGE q. Sketch $\log J$ vs $\log V$, FOR BOTH A NON-RELATIVISTIC DIODE AND AN EXTREME RELATIVISTIC DIODE. AT WHAT VALUE OF qV/mc^2 DO THE CURVES INTERSECT?



(HINT: $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ & $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$ are both RELATIVISTICALLY CORRECT EQUATIONS).

Problem #2
20 points

(2) CONSIDER A HEAVY ION ACCELERATOR WITH CONSTANT AVERAGE BEAM RADIUS r_b, CONSTANT PIPE RADIUS r_p, CONSTANT UNDEPRESSED PHASE ADVANCE σ_0, AND CONSTANT ELECTRIC QUADRUPOLE VOLTAGE V_0, AND CONSTANT QUADRUPOLE OCCUPANCY η. ASSUME THAT THE line charge λ IS CONSTANT OVER A BUNCH OF LENGTH l_b, but can vary as the energy of the beam changes.

Assume that the bunch length is allowed to expand or contract as the beam accelerates such that the line charge density is at the maximum transportable line charge. (Note that the total charge in the bunch is constant).

a) How does the bunch length scale with the ion energy qV? (That is calculate α in the relation $l_b = l_{b0}\left(\frac{V}{V_0}\right)^{\alpha}$ where l_{b0} and V_0 are the values of l_b and V at the beginning of the accelerator.

b) How does $\frac{\Delta p}{p}$ scale with V? (i.e. calculate α_1 in the relation $\frac{\Delta p}{p} = \left(\frac{\Delta p}{p_0}\right)_0 \left(\frac{V}{V_0}\right)^{\alpha_1}$) Assume constant normalized longitudinal emittance. [$\frac{\Delta p}{p}$ = fractional momentum spread].

c) REPEAT QUESTION a) using magnetic quads with constant magnetic gradient B_q/r_p.

d) Repeat QUESTION b) using magnetic quads with constant magnetic gradient B_q/r_p.

Problem #3
15 points

3

a) EXPRESS THE DEBYE LENGTH λ_D OF A NON-RELATIVISTIC, UNIFORM DENSITY BEAM IN TERMS OF BEAM RADIUS r_b, EMITTANCE ε AND PERVEANCE Q.

b). A SPACE CHARGE DOMINATED BEAM IS DEFINED AS A BEAM FOR WHICH THE PERVEANCE TERM IN THE ENVELOPE EQUATION IS MUCH GREATER THAN THE EMITTANCE TERM.

USING PART a) SHOW THAT FOR A SPACE CHARGE DOMINATED BEAM $\lambda_D \ll r_b$.

Problem # 4
15 points

4

LET THE SINGLE PARTICLE EQUATION OF MOTION BE:

$$\frac{d^2x}{ds^2} = -k^2 x$$

HERE k is a constant, x is the usual transverse coordinate, and s is the longitudinal coordinate. Let the initial value of $\langle x^2 \rangle = \langle x_0^2 \rangle$.

What are the values of $\langle x_0 x_0' \rangle$ and $\langle x_0'^2 \rangle$ for which $\frac{d}{ds}\langle x^2 \rangle$, $\frac{d}{ds}\langle x x' \rangle$, and $\frac{d}{ds}\langle x'^2 \rangle$ are all zero?

HERE $\langle\ \rangle$ denotes average over the distribution function and subscript 0 indicates initial value.

(THESE ARE THE CONDITIONS FOR A MATCHED BEAM).

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Particle Orbit in a Solenoidal Magnetic Field

a/ Show that for a constant magnetic field $B_{z0} = \text{Const}$, no acceleration ($\gamma_b \beta_b = \text{const}$), negligible space-charge ($\phi \simeq 0$), that the particle equations of motion derived in class reduce to

$$x'' = \frac{B_{z0}}{[B\rho]} y'$$

$$y'' = -\frac{B_{z0}}{[B\rho]} x'$$

$$[B\rho] = \frac{m\gamma_b \beta_b c}{q}$$

b/ Define a complex coordinate $\underline{z} = x + iy$. Show that the equation of motion in a/ reduces to

$$\underline{z}'' + i k \underline{z}' = 0 \qquad k = \frac{B_{z0}}{[B\rho]} \qquad i = \sqrt{-1}$$

c/ Show that the equation in b/ has two linearly independent solutions:

1) $\underline{z} = \text{const}$

2) $\underline{z} = \text{const}_1 \cdot e^{\text{const}_2 \cdot s}$

Identify const_2 and apply these to derive the general solution to the equation of motion:

$s = s_i$ = initial s

$$\underline{z} = \underline{z}_i - \frac{i\underline{z}_i'}{k} + \frac{i\underline{z}_i'}{k} e^{-ik(s-s_i)}$$

$$\underline{z}' = \underline{z}_i' e^{-ik(s-s_i)}$$

$\underline{z}_i = x_i + i y_i \qquad x_i = x(s=s_i)$, $y_i = y(s=s_i)$

$\underline{z}_i' = x_i' + i y_i' \qquad x_i' = x'(s=s_i)$, $y_i' = y'(s=s_i)$

Why is there only one sign in the exponent? That is, why is $\text{const} \cdot e^{+ik(s-s_i)}$ not also a solution?

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d/ Show from the results of part c/ that the x-y particle orbit is circular. Identify the center and radius of the orbit. Sketch the orbit.

e/ Use the result of part d/ to derive a 4x4 transfer matrix $\bar{M}(s|s_i)$ for the coupled motion:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \bar{M}(s|s_i) \cdot \begin{pmatrix} x_i \\ x_i' \\ y_i \\ y_i' \end{pmatrix}$$

$x_i \equiv x(s=s_i)$

Initial cord at $s=s_i$

8 Problem - Courant Snyder Invariant

As derived in class, a coasting uniform density elliptical beam with $(\gamma_b \beta_b)' = 0$ has particle equations of motion in the beam given by:

$$x'' + \kappa_x(s)\, x - \frac{2Q\, x}{(r_x + r_y)\, r_x} = 0$$

$$y'' + \kappa_y(s)\, y - \frac{2Q\, y}{(r_x + r_y)\, r_y} = 0$$

where r_x and r_y obey the envelope equations:

$$r_x'' + \kappa_x(s)\, r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y(s)\, r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

with

$\varepsilon_x = \text{const}$ $\qquad Q = \dfrac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2}$ = Perveance = const

$\varepsilon_y = \text{const}$

κ_x, κ_y = x- and y-focusing forces (specified in s)

A/ Take a "Phase-Amplitude" form of the particle x-orbit with

$$x = A_x(s) \cos \psi_x(s)$$

A solution of this form is known to exist by Floquet's theorem. Taking this for granted, show that the x equation of motion is then equivalent to two equations:

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$$A_x'' + \kappa_x A_x - \frac{2Q A_x}{(r_x+r_y)r_x} - A_x \psi_x'^2 = 0 \qquad -(1)$$

$$A_x \psi_x'' + 2A_x'\psi_x' = 0 \qquad -(2)$$

B/ Show that Eq. (2) in part A/ has a solution

$$\psi_x' = \frac{C}{A_x^2} \qquad C = \text{const.}$$

and show that if we take

$$C = a^2 \varepsilon_x \qquad a = \text{const}$$

$$A_x = a\, r_x \qquad (\text{dimensionless amplitude})$$

that the particle orbit is consistent with the x-equation for the beam envelope:

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

C/ From the results of part B/, the particle orbit in the beam can be expressed as:

$$x = a\, r_x \cos \psi_x$$

Show that the particle orbit has a single-particle invariant of the form:

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 = a^2 = \text{const.}$$

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This Courant-Snyder invariant is the equation of an ellipse in x-x' phase-space.

D/ Note that ψ_x satisfies:

$$\psi_x' = \frac{C}{A_x^2} = \frac{\varepsilon_x}{r_x^2}$$

independent of a. Thus ψ_x is independent of particle amplitude and we expect the amplitudes of particle orbits of the uniform density beam to be uniformly distributed with $0 \le a \le 1$. Use this and the invariant in part C to show that the maximum particle orbits define an ellipse with

$$\text{Area} = \pi \varepsilon_x$$

in x-x' phase space.

Hint: The rotated ellipse:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 1$$

has area

$$\text{Area} = \frac{\pi}{\sqrt{\gamma\beta - \alpha^2}},$$

These results reinforce that the statistical emittance

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle x x'\rangle_\perp^2\right]^{1/2}$$

is π × the x-x' phase-space area of the maximum particle orbits in a KV beam and that all particles move on nested ellipses in x-x' phase-space.

TED Problem 9

Thermal Equilibrium — Problem # 7, 15 points — S. M. Lund — P9/

9/ In 3D a thermal equilibrium can be constructed with $\partial/\partial z = 0$ (unbunched) that is a straightforward generalization of the cylindrical equilibrium presented in class. There is essentially one additional Gaussian integral over the longitudinal beam-frame momentum.

Assume a nonrelativistic beam ($\gamma_b = 1$). Following the procedure in class, a test charge q_T is placed at the origin of the 3D thermal equilibrium beam. With analogous approximations a 3D Poisson equation valid in the beam core can be derived:

$$\nabla^2 \delta\phi - \frac{\delta\phi}{\lambda_D^2} = -\frac{q_T}{\varepsilon_0}\delta(\vec{x})$$

where:

$$\delta(\vec{x}) = \delta(x)\delta(y)\delta(z)$$

$$\nabla^2 = \frac{\partial}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{x}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Show that this equation has a solution regular at infinity satisfying

$$\delta\phi = \frac{q_T}{4\pi\varepsilon_0 r} e^{-r/\lambda_D} \qquad r = \sqrt{x^2+y^2+z^2}$$

Hints: You can construct the solution by

1) matching near and far solutions as in class notes — see: Transverse Equilibrium Distribution Notes.
2) In 3D spherically symmetric geometry
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \qquad r = \sqrt{x^2+y^2+z^2}$$
3) Try transforming the equation using $\delta\phi = \tilde{\phi}/r$ and solving for $\tilde{\phi}$.

TCE Problem 1 — Problem #8, 10 points — S.M. Lund P1/

1/ ## Envelope Radius of a Nonuniform Density Beam

For a uniform density elliptical beam with envelope radii r_x and r_y

$$n(x,y) = \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} & ; \quad \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

λ = line-charge density = const.

and we showed in previous problems that

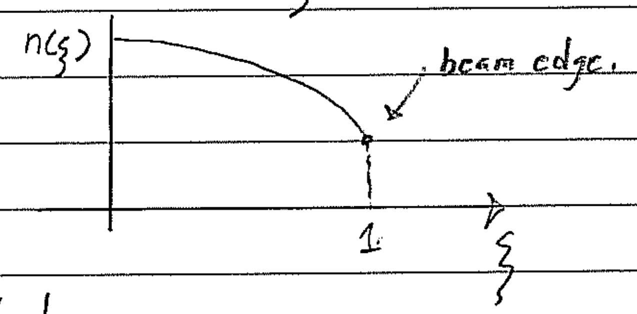
$$r_x = 2\langle x^2\rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2\rangle_\perp^{1/2}$$

where $$\langle x^2\rangle = \frac{\int d^2x \; x^2 n(x,y)}{\int d^2x \; n(x,y)}$$

If the density profile is elliptical with

$$n(x,y) = n(\xi) \quad ; \quad \xi^2 \equiv \frac{x^2}{r_{xe}^2} + \frac{y^2}{r_{ye}^2} \qquad \begin{aligned} r_{xe} &= \text{const} \\ r_{ye} &= \text{const} \end{aligned}$$

such that $n(\xi)$ is monotonic decreasing in ξ with a sharp cutoff at $\xi = 1$



Show that

$$r_{xe} > r_x \equiv 2\langle x^2\rangle_\perp^{1/2}$$
$$r_{ye} > r_y \equiv 2\langle y^2\rangle_\perp^{1/2}$$

where $\langle x^2\rangle_\perp$ and $\langle y^2\rangle_\perp$ are defined from the nonuniform density beam. You may use steps/transforms from previous problems