①

John Barnard
Steven Lund
NE-290H
Spring 2009
U.C. Berkeley

I. <u>Introduction</u>

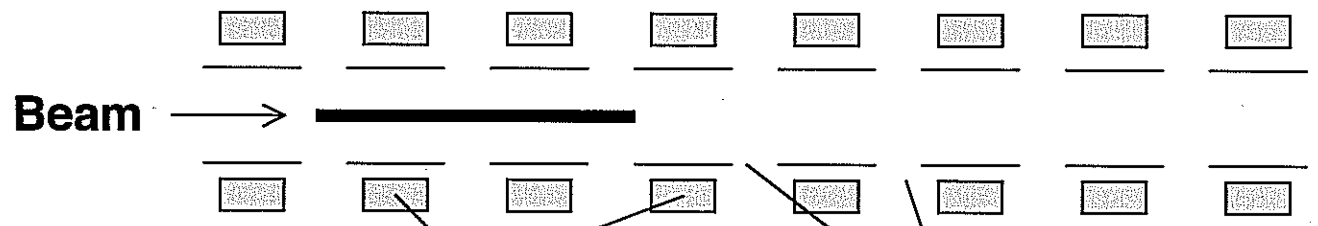
(related reading in parentheses)

Particle motion (Reiser 2.1)
Equation of motion (Reiser 2.1)
Dimensionless quantities (Reiser 4.2)

Plasma physics of beams (Reiser 3.2, 4.1)

Emittance and brightness (Reiser 3.1 - 3.2)

How do we describe and calculate the evolution of a collection of particles under the EM forces in an accelerator?

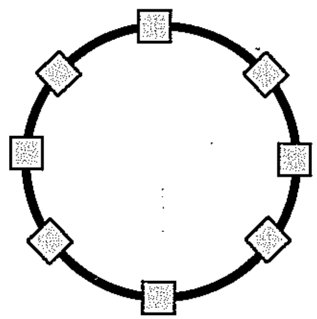


Electric or magnetic elements that confine or focus the beam transversely

Electric fields typically in gaps for longitudinal acceleration and focusing

This array or "lattice" of focusing elements may be arranged in a linac or circular accelerator

or

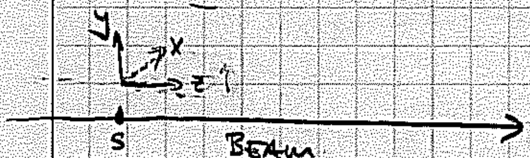


PARTICLE EQUATIONS OF MOTION / DIMENSIONLESS QUANTITIES

CONSIDER THE LORENTZ FORCE ON A PARTICLE UNDER THE INFLUENCE OF ELECTRIC AND MAGNETIC FORCES.

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B}) \qquad [\text{SI units}]$$

$$\underline{p} = \gamma m \underline{v} \qquad \gamma^2 = \frac{1}{1-\beta^2} \qquad \beta = \frac{v}{c}$$



CONSIDER THE x-COMPONENT OF THE MOTION (TRANSVERSE TO THE STREAMING MOTION OF THE PARTICLE)

TRANSFORM TO s AS THE INDEPENDENT VARIABLE.

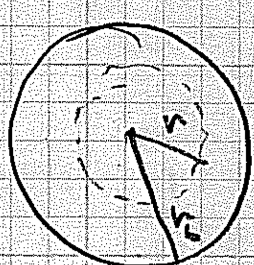
$$dt = \frac{ds}{v_z} \quad \Rightarrow \quad v_x = \frac{dx}{dt} = v_z x' \qquad {}' = \frac{d}{ds}$$

$$m v_z \frac{d}{ds}(\gamma v_z x') = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\gamma m v_z^2 x'' + x' m v_z \frac{d}{ds}(\gamma v_z) = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\right] x' = \frac{q}{\gamma m v_z^2}(\underline{E} + \underline{v} \times \underline{B})_x$$

NOW CONSIDER AN UNBUNCHED BEAM OF UNIFORM DENSITY ρ AND CIRCULAR CROSS SECTION



$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0}$$

$$2\pi r E_r = \pi r^2 \frac{\rho}{\epsilon_0} \qquad \text{(Gauss' theorem)}$$

$$\lambda = \pi r_b^2 \rho$$

$$E_r = \frac{\rho}{2\epsilon_0} r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{r_b^2} \qquad E_x = E_r\cos\theta = E_r\left(\frac{x}{r}\right) = \frac{\lambda}{2\pi\epsilon_0}\frac{x}{r_b^2}$$

Similarly $\underline{\nabla}\times\underline{B} = \mu_0 \underline{J}$

$$2\pi r B_\theta = \mu_0 \rho v_z \pi r^2 \qquad \text{(Stokes theorem)}$$

$$B_\theta = \frac{\mu_0 \lambda v_z}{2\pi}\frac{r}{r_b^2} \qquad B_y = \frac{\mu_0 \lambda v_z}{2\pi}\frac{x}{r_b^2} \qquad (B_z = 0)$$

Let $(\underline{E} + \underline{v}\times\underline{B})_x = (E_x - v_z B_y)^{self} + (E_x + v_y B_z - v_z B_y)^{ext}$

$$\Rightarrow \quad x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}\gamma v_z\right]x' = \frac{q}{\gamma m v_z^2}\frac{\lambda}{2\pi\epsilon_0}\frac{x}{r_b^2}\left[1 - \mu_0\epsilon_0 v_z^2\right] + \frac{q}{\gamma m v_z^2}(\underline{E}+\underline{v}\times\underline{B})_x^{ext}$$

USING $\mu_0\epsilon_0 = \frac{1}{c^2}$

Assuming $\beta_x^2 + \beta_y^2 << \frac{1}{\gamma^2} \Rightarrow \gamma^2 \simeq \frac{1}{1 - v_z^2/c^2}$ (PARAXIAL APPROXIMATION)

($\Rightarrow \tilde{\beta}_x^2 + \tilde{\beta}_y^2 << 1$; HERE ~ INDICATES VALUE IN COMOVING FRAME [NON-RELATIVISTIC MOTION IN COMOVING FRAME].)

LUMPING EXTERNAL FORCE INTO A LINEAR FIELD

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\, x' \simeq \frac{q}{\gamma^3 m v_z^2}\frac{\lambda}{2\pi\epsilon_0}\frac{x}{r_b^2} - \underbrace{K(z)x}_{\text{EXTERNAL FORCES}}$$

$$\equiv Q\frac{x}{r_b^2} - K(z)\,x$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 \gamma^3 m v_z^2} = \text{GENERALIZED PERVEANCE} \rightarrow \begin{cases} \dfrac{\lambda}{4\pi\epsilon_0 V} & \text{for } \gamma^2 v_z^2 \ll c \\[2ex] \dfrac{\lambda}{2\pi\epsilon_0 V\left(\frac{qV}{mc^2}\right)^2} & \text{for } \gamma^2 v_z^2 \gg \end{cases}$$

$$= \frac{(q/e)}{(m/m_{amu})}\frac{2I}{I_0}\frac{1}{\gamma^3\beta^3} \qquad I_0 = \frac{4\pi\epsilon_0 m_{amu} c^3}{e} \simeq 31\ \text{MA}$$

here $qV \equiv (\gamma - 1)mc^2$

Also note in non-relativistic limit $Q = \left(\frac{m}{2q}\right)^{1/2}\frac{1}{4\pi\epsilon_0}\left(\frac{I}{V^{3/2}}\right)$

(same scaling as original term perveance characterizing injectors).

$$Q \simeq \frac{\Phi_{SELF}}{V} = \frac{\int_0^{r_b}(E_r - v_z B_\theta)\,dr}{V} = \frac{\text{POTENTIAL ENERGY OF BEAM PARTICLE}}{\text{KINETIC ENERGY OF " "}}$$

SOMETIMES PERIODIC FOCUSING IS EMPLOYED

$$K(z) = K(z+S)$$

$$S = \text{PERIOD}$$

FOR SOME PURPOSES A SUITABLE CONSTANT CAN BE FOUND WHICH CAPTURES SLOW VARIATION OF THE PARTICLE MOTION. (SMOOTH FOCUSING APPROX.)

$$\Rightarrow \boxed{x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\,x' = Q\frac{x}{r_b^2} - k_{\beta 0}^2 x}$$

$k_{\beta 0} \equiv$ "UNDEPRESSED BETATRON FREQUENCY" $\qquad \sigma_0 \equiv k_{\beta 0} S$ = UNDEPRESSED PHASE ADVANCE

If $\frac{dv_z}{dz} = 0$ [drifting beams]

$$x'' = -\left[k_{\beta 0}^2 - \frac{Q}{r_b^2} \right] x$$

$$= -k_{\beta 0}^2 \underbrace{\left[1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right]}_{\equiv \left(\frac{\sigma}{\sigma_0}\right)^2 = (\text{"TUNE DEPRESSION"})^2} x \equiv -k_\beta^2 x$$

"DEPRESSED BETATRON FREQUENCY"

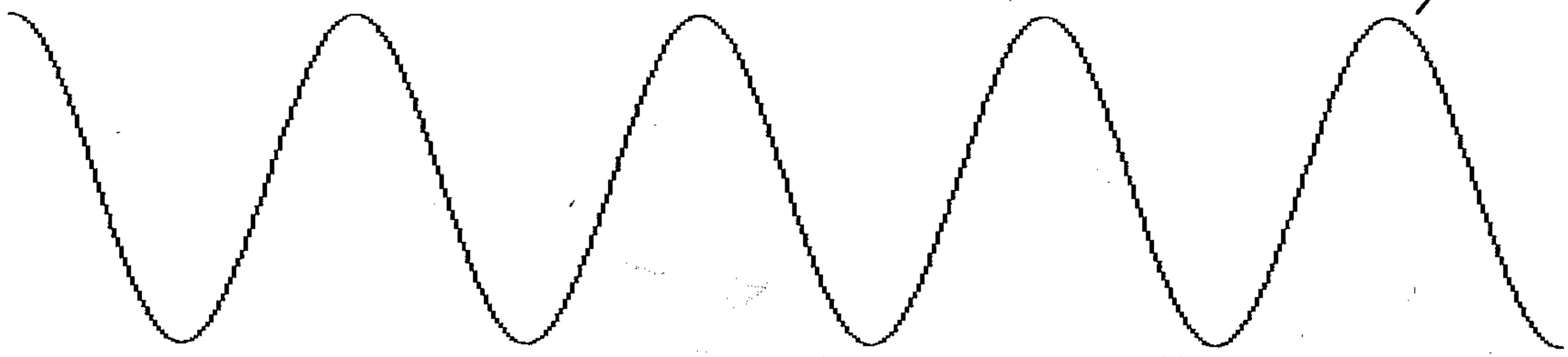
EFFECT OF SPACE CHARGE IS TO LOWER FREQUENCY OF HARMONIC OSCILLATIONS

$\frac{\sigma}{\sigma_0} = 0 \Rightarrow$ FULLY TUNE DEPRESSED

$\frac{\sigma}{\sigma_0} = 1 \Rightarrow$ NO SPACE-CHARGE DEPRESSION

Space charge reduces betatron phase advance

Without space charge: $x = x_i \cos\left[k_{\beta 0}(s - s_i)\right] + \frac{x_i'}{k_{\beta 0}} \sin\left[k_{\beta 0}(s - s_i)\right]$



With space charge:

$\sigma/\sigma_0 \sim 5/18 \sim 0.277$

$x = x_i \cos\left[k_{\beta 0}\frac{\sigma}{\sigma_0}(s - s_i)\right] + \frac{x_i'}{\left(\frac{\sigma}{\sigma_0}\right)k_{\beta 0}} \sin\left[k_{\beta 0}\frac{\sigma}{\sigma_0}(s - s_i)\right]$

J. BARNARD

BENDING BEAMS

RETURNING TO PARTICLE EQUATION WITH ARBITRARY $E, \& B$:

$$x'' + \left[\frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\right] x' = \frac{q}{\gamma m v_z^2}\left(\underline{E} + \underline{v}\times\underline{B}\right)_x$$

IF EXTERNAL FORCE IS PROPORTIONAL TO $-x$

$\Rightarrow$ FOCUSING (HARMONIC OSCILLATIONS)

HOWEVER, IF $\underline{E} + \underline{v}\times\underline{B}$ = CONSTANT

$\Rightarrow$ BENDING

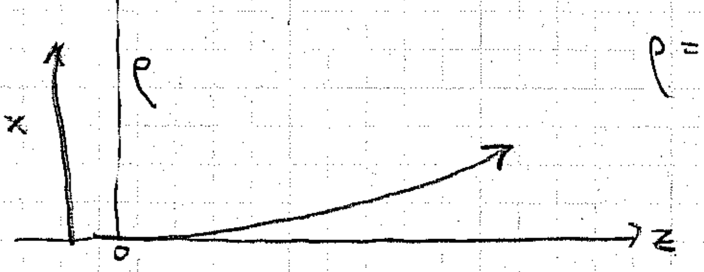
EXAMPLE: If $B = B_y \hat{e}_y$

$v = v_0 \hat{e}_z + v_x \hat{e}_x$ where $v_0 \gg v_x$

$$\Rightarrow \quad x'' = \frac{qB_y}{\gamma m v_z^2} = \frac{B_y}{[B\rho]} \qquad [B\rho] \equiv \text{RIGIDITY} = \frac{\gamma m v_z}{q} = \frac{p}{q}$$

$$x' = \frac{B_y}{[B\rho]} z + x_0'$$

$$x = \frac{B_y}{[B\rho]}\frac{z^2}{2} + x_0' z + x_0$$



$$\rho = \text{RADIUS OF CURVATURE OF ARC} = \frac{[B\rho]}{B_y}$$

(BENDING CAN ALSO BE CARRIED OUT WITH ELECTRIC FIELDS E = [illegible])

PLASMA PHYSICS OF BEAMS

PHYSICS OF SPACE CHARGE ≡ PHYSICS OF SELF FIELDS
≡ PLASMA PHYSICS OF PARTICLE BEAMS

PLASMA PARAMETER Λ

$$q\Phi_{IP} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r_{IP}}$$

AVERAGE POTENTIAL ENERGY $q\Phi_{IP}$ OF PARTICLE DUE TO ITS NEAREST NEIGHBOR A DISTANCE r_{IP}
(q = CHARGE OF PARTICLE)

$$\simeq \frac{1}{4\pi\epsilon_0} n_0^{1/3} q^2$$

n_0 = number density

IF $q\Phi_{IP} \ll k_B T$ $\Rightarrow$ WEAKLY COUPLED PLASMA OR SIMPLY PLASMA

DEFINE $$\lambda_D \equiv \frac{(k_B T/m)^{1/2}}{(n_0 q^2/\varepsilon_0 m)^{1/2}} = \frac{V_{TH}}{\omega_p} = \left(\frac{k_B T\,\varepsilon_0}{n_0 q^2}\right)^{1/2}$$ DEBYE LENGTH

= CHARACTERISTIC DISTANCE WHEREBY CHARGES ARE SHIELDED

DEFINE $\Lambda = \frac{4\pi}{3} n_0 \lambda_D^3$ ≡ PLASMA PARAMETER

$$\sim \left(\frac{k_B T}{q\Phi_{IP}}\right)^{3/2} \gg 1 \quad [\text{if } q\Phi_{IP} \ll k_B T]$$

J. BARNARD (10)

REF. "INTRO. TO PLASMA THEORY", D.R. NICHOLSON, WILEY, 1983.

KLIMONTOVICH EQUATION

let $N(\underline{x},\underline{v},t) = \sum_{i=1}^{N_0} \delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$

N_0 particles; $\underline{X}_i, \underline{V}_i$ are position and velocity of ith particle

$\dot{\underline{X}}_i = \underline{V}_i \qquad m\dot{\underline{V}}_i = q\underline{E}^m[\underline{X}_i(t),t] + q[\underline{V}_i \times B^m[\underline{X}_i(t),t]]$ (non-relativistic)

$N(x,v,t)$ = "density" of particles in phase space

$$\int N\, d^3x\, d^3v = N_0$$

let $u = \underline{x} - \underline{X}_i(t)$

$\frac{\partial}{\partial x} f(u) = f'(u)$

$\frac{\partial}{\partial t} f(u) = f'(u)(-\dot{X}(t))$

$= -\dot{X}(t)\frac{\partial}{\partial x} f(u)$

Taking derivatives:

$$\frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N} \dot{\underline{x}}_i(t)\cdot\nabla_x\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$
$$-\sum_{i=1}^{N_0} \dot{\underline{V}}_i(t)\cdot\nabla_v\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$

$$\underline{\nabla}\cdot\underline{E}^m = \overbrace{\left(\frac{1}{\varepsilon_0}\right) q\int dv\, N(x,v,t)}^{\rho^m} \qquad \underline{\nabla}\cdot\underline{B}^m = 0$$

$$\nabla\times\underline{E}^m = -\frac{\partial B^m}{\partial t} \qquad \nabla\times\underline{B}^m = \underbrace{\mu_0 q\int d\underline{v}\,\underline{v}\, N(x,v,t)}_{J^m} + \frac{\partial \underline{E}^m}{\partial t}$$

$$\Rightarrow \frac{\partial N}{\partial t}(x,v,t) = -\sum_{i=1}^{N_0} \underline{V}_i(t)\cdot\nabla_x\left[\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))\right]$$
$$-\sum_{i=1}^{N_0}\left(\left(\frac{q}{m}\right)E^m + \left(\frac{q}{m}\right)[\underline{V}_i\times B^m[X_i(t),t]]\right)\cdot\nabla_v\left[\delta(x-X_i)\delta(v-V_i(t))\right]$$

Note that $\underline{V}_i(t)\,\delta(\underline{v}-\underline{V}_i(t)) = \underline{v}\,\delta(\underline{v}-V_i(t))$ so,

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x},\underline{v},t) = -\underline{v}\cdot\nabla_x\sum_{i=1}^{N_0}\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$
$$-\left(\frac{q}{m}E^m(x,t) + \frac{q}{m}(\underline{v}\times\underline{B}^m(x,t))\right)\cdot\nabla_v\sum_{i=1}^{N_0}\delta(x-\underline{X}_i(t))\delta(v-V_i(t))$$

$$\boxed{\frac{\partial N}{\partial t}(x,v,t) = -\underline{v}\cdot\nabla_x N(x,v,t) + \frac{q}{m}(E^m + v\times B^m)\cdot\nabla_v N(\underline{x},\underline{v},t)}$$

Klimontovich Equation

Total derivative along an orbit:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left.\frac{d\underline{x}}{dt}\right|_{\text{ORBIT}} \cdot \nabla_x + \left.\frac{d\underline{v}}{dt}\right|_{\text{ORBIT}} \cdot \nabla_v$$

$$\Rightarrow \boxed{\frac{D}{Dt} N(\underline{x}, \underline{v}, t) = 0}$$

Note that $N = 0$ or ∞, nothing in between.

$$\text{Let} \quad f(x, v, t) = \frac{\int^{\Delta x^3, \Delta v^3} N(\underline{x}, \underline{v}, t)\, d^3x\, d^3v}{\Delta x^3 \Delta v^3}$$

over some box in phase space
Δx & Δv are the size of box

$$\equiv \langle N(\underline{x}, \underline{v}, t) \rangle$$

Assume $n^{-1/3} \ll \Delta x \ll \lambda_D$
so that $f(x, v, t)$ is smooth function.

Then

$$N = f + \delta f \qquad f = \langle N \rangle \qquad \langle \delta f \rangle = 0$$
$$E^m = E + \delta E \qquad E = \langle E^m \rangle \qquad \langle \delta E \rangle = 0$$
$$B^m = B + \delta B \qquad B = \langle B^m \rangle \qquad \langle \delta B \rangle = 0$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}) \cdot \nabla_v f}_{\text{SMOOTHLY VARYING PART}} = \underbrace{-\frac{q}{m}\left\langle (\delta \underline{E} + \underline{v} \times \delta \underline{B}) \cdot \nabla_v \delta f \right\rangle}_{\text{AVERAGE OF "SPIKEY" QUANTITIES}}$$

"DISCRETE PARTICLE EFFECTS" OR "COLLISIONS"

If collisions are neglected (set RHS to zero):

VLASOV - EQUATION

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_x f + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{b}) \cdot \nabla_v f = 0$$

$$\Rightarrow \boxed{\frac{Df}{Dt} = 0}$$

PHASE SPACE DENSITY CONSTANT ON TRAJECTORIES. (LIOUVILLE'S THEOREM)

THE RHS IS DUE TO COLLISIONS WITH NON-SMOOTH FIELDS:

VERY HEURISTICALLY

$$-\frac{q}{m}\left\langle \left(\delta \underline{E} + \underline{v} \times \delta \underline{B}\right) \cdot \nabla_v \delta f \right\rangle \sim \nu_c f$$

$\nu_c \sim \sigma n v$

$\sigma \sim \pi r_c^2$ where r_c is given by $kT \sim \dfrac{q^2}{4\pi\epsilon_0 r_c}$

(for large angle collisions)

$$\Rightarrow \nu_c \sim \pi \left(\frac{q^2}{4\pi\epsilon_0 kT}\right)^2 n_0 \left(\frac{kT}{m}\right)^{1/2} \sim \frac{1}{16\pi}\frac{v_{th}}{\lambda_D^4 n_0}$$

$\longleftarrow$ (VERY ROUGH, BUT MAIN SCALING IS CORRECT, WITH LOGARITHMIC CORRECTION FACTORS)

ON LHS OF VLASOV EQUATION:

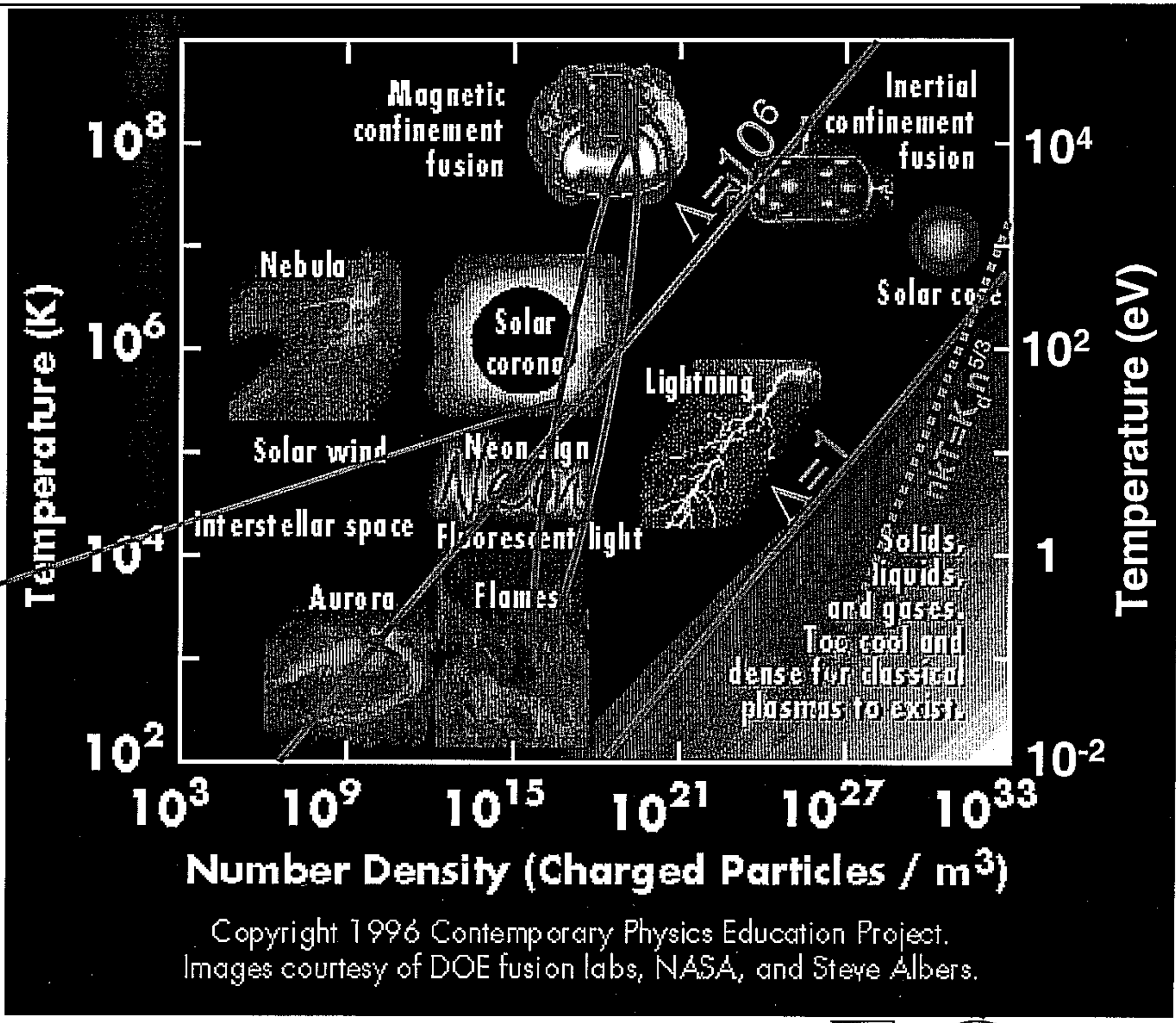
$$\frac{q}{m} E \nabla_v f \sim \frac{q}{m}\left(\frac{q \lambda_D n_0}{\epsilon_0}\right)\frac{f}{v_{TH}} \sim \omega_p f$$

where $v_{TH} \sim \sqrt{\dfrac{kT}{m}}$

$$\frac{\text{COLLISION TERM}}{\text{LHS}} \sim \frac{1}{16\pi \lambda_D^3 n_0} = \frac{1}{16\Lambda}$$

Accelerator beams are non-neutral plasmas

Accelerator beams for Heavy Ion Fusion



J. BARNARD

DESCRIPTION OF THE BEAM

LIOUVILLE'S THEOREM: $\frac{df}{dt} = 0$ along a trajectory in phase space.

Let $dN = f\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z$

The continuity equation in phase space is:

$$\frac{\partial f}{\partial t} + \underline{\nabla} \cdot (f \underline{v}) = 0$$

where $\underline{v} = \frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ & $\underline{\nabla} \cdot \underline{a} = \frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2} + \frac{\partial a_3}{\partial q_3} + \frac{\partial a_{p_1}}{\partial p_1} + \frac{\partial a_{p_2}}{\partial p_2} + \frac{\partial a_{p_3}}{\partial p_3}$

($\underline{v}$ & $\underline{\nabla}\cdot$ are the 6-D velocity & divergence, respectively).

If the system is governed by a Hamiltonian $H(\underline{q}, \underline{p}, t)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{-\partial H}{\partial q_i}$$

Now,
$$\underline{\nabla} \cdot \underline{v} = \sum_{i=1}^{3} \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \sum_{i=1}^{3} \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + f \underbrace{\cancel{\underline{\nabla} \cdot \underline{v}}}_{0} + \underline{v} \cdot \nabla f = 0$$

$$\Rightarrow \boxed{\left.\frac{df}{dt}\right|_{\substack{6D \\ \text{trajectory}}} = 0}$$

TOPS FORM 3304 1

Emittance & Brightness

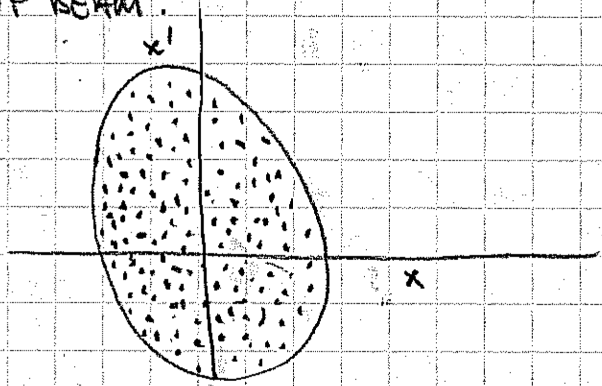
LIOUVILLE'S EQUATION OR VLASOV EQUATION $\Rightarrow \dfrac{dN}{dx\,dy\,dz\,dp_x\,dp_y\,dp_z} = \text{const}$

IF $x'' = f(x)$ AND NOT FUNCTIONS $(y \,\&\, z)$
$y'' = f(y) =$ " " " $(x, \,\&\, z)$
$z'' = f(z) =$ " " " $(x, \,\&\, y)$

THEN $\dfrac{dN}{dx\,dp_x} = \text{const}$; $\dfrac{dN}{dy\,dp_y} = \text{const}$ & $\dfrac{dN}{dz\,dp_z} = \text{const}$

separately.

1ST DEFINITION:
EMITTANCE: USE TRACE-SPACE OF ALL PARTICLES IN A GIVEN SLICE OF BEAM.



INSTEAD OF p_x USE $x' = \dfrac{v_x}{v_z}$ (FOR NON-ACCELERATING PARAXIAL BEAM, x' PROPORTIONAL TO MOMENTUM)

EMITTANCE $= \dfrac{1}{\pi} \cdot$ AREA OF SMALLEST ELLIPSE WHICH ENCLOSES ALL PARTICLES. (TRACE-SPACE DEFINITION)

(INTUITIVELY, PRODUCT OF WIDTH IN x, TIMES WIDTH IN x', SO IT IS ESSENTIALLY (WITHIN FACTOR OF π) = PHASE SPACE AREA OF BEAM.

2ND DEFINITION INVOLVES STATISTICAL AVERAGES OF 2ND ORDER QUANTITIES (SUCH AS rms).

$$\epsilon_x \equiv 4\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)^{1/2}$$

For an upright uniform beam (in phase space): $\langle x^2\rangle = \dfrac{r_x^2}{4}$ $\langle x'^2\rangle = \dfrac{x'^2_{max}}{4}$

& $\langle xx'\rangle = 0$

$$\Rightarrow \epsilon_x = r_x x'_{max} = \frac{\text{Area}}{\pi}$$

NORMALIZED EMITTANCE

For a beam that is accelerating, return to x, p_x as definition of phase space area:

$$p_x = \gamma m v_x = \gamma m v_z x' \qquad \text{AGAIN, assuming } v \simeq v_z$$

$$\Rightarrow \epsilon_{Nx} \equiv 4\gamma\beta \left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right)^{1/2} = \gamma\beta\, \epsilon_x$$

$$= \frac{4}{m} \left(\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 \right)^{1/2}$$

SINCE EMITTANCE IS THE AVERAGE PHASE SPACE AREA OF BEAM (AVERAGING OVER EMPTY SPACE) THE EMITTANCE IN GENERAL GROWS AS A BEAM FILAMENTS (ENGULFING EMPTY SPACE).

BRIGHTNESS

THE DENSITY OF PARTICLES IN 6-D PHASE SPACE IS:

$$\frac{dN}{dx\, dy\, dz\, dp_x\, dp_y\, dp_z} = f \quad \longleftarrow \text{MICROSCOPIC DENSITY}$$

DEFINE A QUANTITY $\bar{f}$ WHICH IS THE PHASE-SPACE DENSITY IN AN AVERAGE SENSE

$$\bar{f} = \left\langle \frac{dN}{dx\, dp_x\, dy\, dp_y\, dz\, dp_z} \right\rangle = \frac{(I\Delta t)/q}{\pi^3 \epsilon_{Nx} \epsilon_{Ny} \epsilon_{Nz}}$$

Note $f(\underline{x}, \underline{p}) =$ constant along a trajectory, whereas $\bar{f}$ usually is a decreasing function of z.

NORMALIZED BRIGHTNESS $\quad B_N \equiv \dfrac{I}{\epsilon_{Nx} \epsilon_{Ny}}$

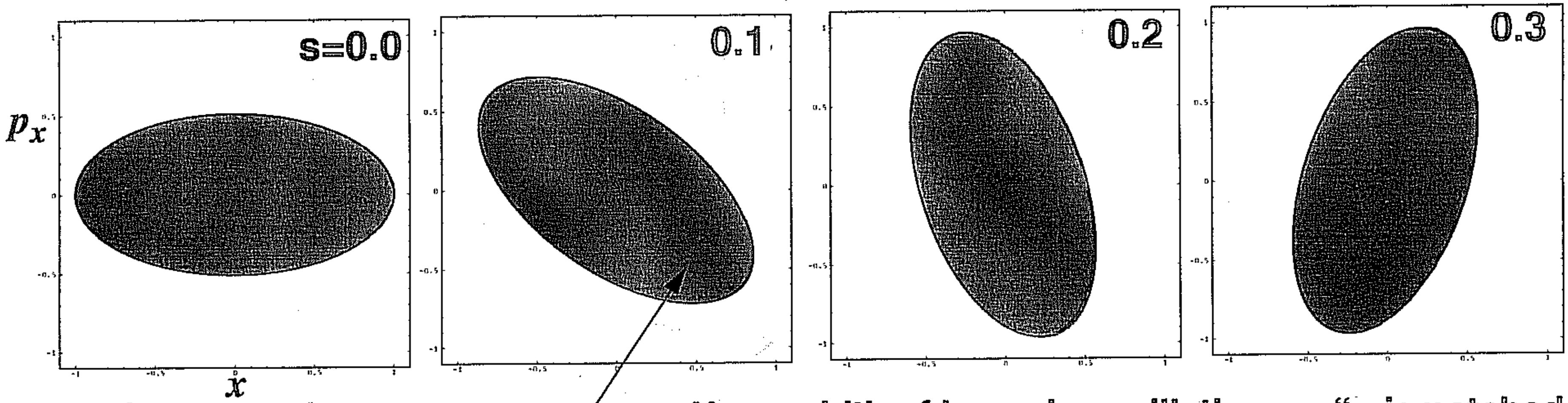
IS A USEFUL MEASURE OF 4D AVERAGE PHASE SPACE DENSITY, (if Δt = constant, & $\perp$,$\parallel$ motion is uncoupled.)

For non-accelerating beams the unnormalized brightness B (also if Δt = const. & $\perp$,$\parallel$ motion uncoupled):

$$\Rightarrow B \equiv \frac{I}{\epsilon_x \epsilon_y} \qquad \text{MEASURES PHASE SPACE DENSITY.}$$

Emittance constant for linear force profile & matched beams

Linear force profile ($x'' = -k^2 x$) => Phase space area preserved, ellipse stays elliptical.



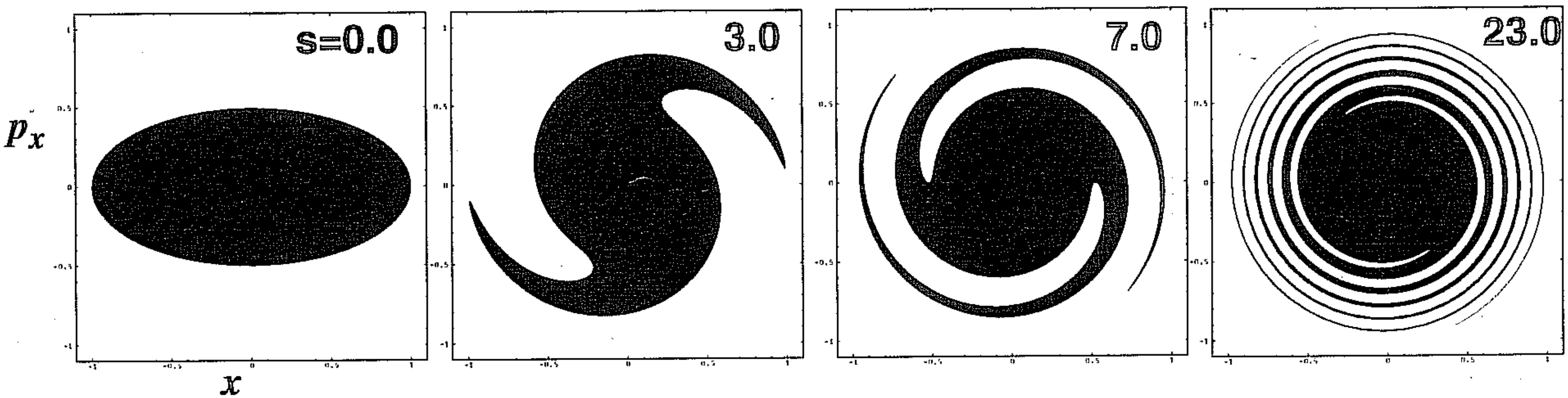
Emittance = phase space area

Emittance constant if forces linear

Here, width of beam is oscillating or "mismatched."

Non-linear forces (e.g. $x'' = -k^2 x + \varepsilon x^3$) $\Rightarrow$ position-dependent frequency

$\Rightarrow$ phase mixing, increasing effective area $\Rightarrow$ Emittance increases if forces non-linear



J. Barnard 17