John Barnard
Steven Lund
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U.C. Berkeley

II. Envelope Equations

Paraxial Ray Equation

Envelope equations for axially symmetric beams

Cartesian equation of motion

Envelope equations for elliptically symmetric beams

2

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Steven Lund
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Roadmap:

Single particle equation with Lorentz force
q(E + v x B)

$\Downarrow$

Make use of:

1. Paraxial (near-axis) approximation
   (Small r and r')
2. Conservation of canonical angular momentum
3. Axisymmetry f(r,z)

$\Downarrow$

Paraxial Ray Equation for Single Particle

Next take statistical averages over the distribution function

$\Rightarrow$ Moment equations

Express some of the moments in terms of the rms radius and emittance

$\Rightarrow$ Envelope equations (axi-symmetric case)

Some focusing systems have quadrupolar symmetry
Rederive envelope equations in cartesian coordinates
(x,y,z) rather than radial (r,z)

START WITH NEWTON'S EQUATION WITH THE LORENTZ FORCE:

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$$

In cartesian coordinate:

$$\frac{d}{dt}(\gamma m\dot{x}) = \dot{\gamma} m\dot{x} + \gamma m\ddot{x} = q(E_x + \dot{y}B_z - \dot{z}B_y)$$

$$\frac{d}{dt}(\gamma m\dot{y}) = \dot{\gamma} m\dot{y} + \gamma m\ddot{y} = q(E_y + \dot{z}B_x - \dot{x}B_z)$$

$$\frac{d}{dt}(\gamma m\dot{z}) = \dot{\gamma} m\dot{z} + \gamma m\ddot{z} = q(E_z + \dot{x}B_y - \dot{y}B_x)$$

In cylindrical coordinates: (use $\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta}$; $\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta}$)
(see next page)

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta) \qquad (\mathrm{I})$$

$$\frac{1}{r}\frac{d}{dt}(\gamma m r^2\dot{\theta}) = q(E_\theta + \dot{z}B_r - \dot{r}B_z) \qquad (\mathrm{II})$$

$$\frac{d}{dt}(\gamma m\dot{z}) = q(E_z + \dot{r}B_\theta - r\dot{\theta}B_r) \qquad (\mathrm{III})$$

When $\frac{\partial}{\partial\theta} = 0$:

$$\underline{E} = -\underline{\nabla}\Phi - \frac{\partial\underline{A}}{\partial t} = \hat{e}_r\left[-\frac{\partial\Phi}{\partial r} - \frac{\partial A_r}{\partial t}\right] + \hat{e}_\theta\left[-\frac{\partial A_\theta}{\partial t}\right] + \hat{e}_z\left[-\frac{\partial\Phi}{\partial z} - \frac{\partial A_z}{\partial t}\right]$$

$$\underline{B} = \nabla\times\underline{A} = \hat{e}_r\left[-\frac{\partial}{\partial z}A_\theta\right] + \hat{e}_\theta\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \hat{e}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right]$$

RHS of II multiplied by r:

$$qr(E_\theta + \dot{z}B_r - \dot{r}B_z) = q\left(-\frac{\partial rA_\theta}{\partial t} - \dot{z}\frac{\partial rA_\theta}{\partial z} - \dot{r}\frac{\partial}{\partial r}(rA_\theta)\right)$$

$$= -q\left[\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial\underline{x}}\right](rA_\theta)$$

$$= -q\frac{d(rA_\theta)}{dt} \qquad (\mathrm{IV})$$

Eqtn II & IV $\Rightarrow$ $\frac{d}{dt}(\gamma m r^2\dot{\theta} + qrA_\theta) = 0$

TO CALCULATE THE RATE OF CHANGE OF THE MOMENTUM $\underline{p}$ IN CYLINDRICAL COORDINATES, WE MUST TAKE INTO ACCOUNT THAT THE UNIT VECTOR CHANGES DIRECTION AS THE PARTICLE MOVES:

LET $$\underline{p} = p_r \hat{e}_r + p_\theta^* \hat{e}_\theta + p_z \hat{e}_z = \gamma m \underline{v}$$

where
$$p_r = \gamma m \dot{r}$$
$$p_\theta^* = \gamma m r \dot{\theta}$$
$$p_z = \gamma m \dot{z}$$

NOTE: ON THIS PAGE $p_\theta^* \equiv \theta$-component of mechanical momentum
NOT TO BE CONFUSED WITH
$p_\theta \equiv \gamma m r^2 \dot{\theta} + q r A_\theta \equiv \theta$-component of CANONICAL ANGULAR momentum

So $$\frac{dp}{dt} = \dot{p}_r \hat{e}_r + p_r \dot{\hat{e}}_r + \dot{p}_\theta^* \hat{e}_\theta + p_\theta^* \dot{\hat{e}}_\theta + \dot{p}_z \hat{e}_z$$

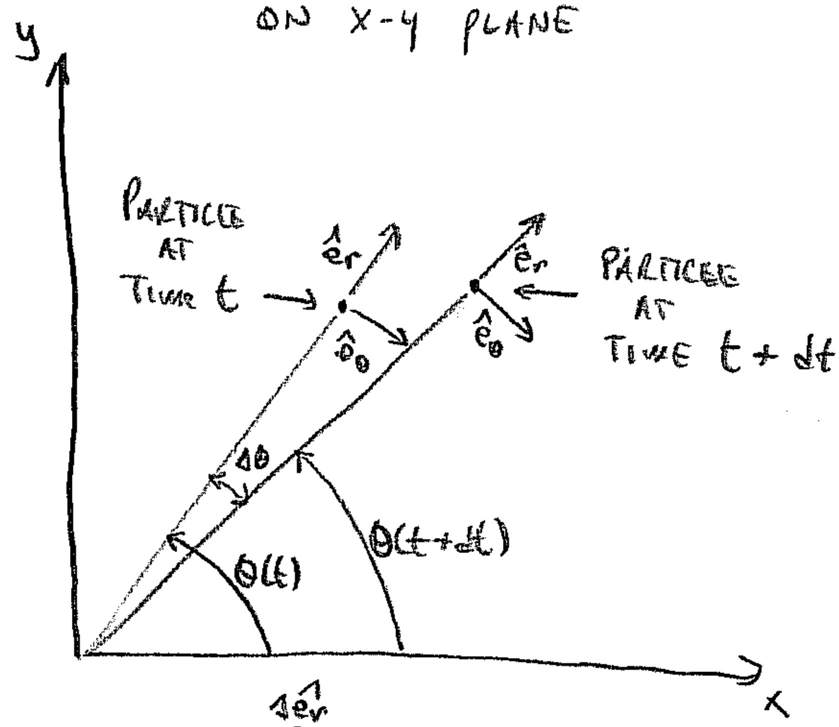
$$\Rightarrow \frac{dp}{dt} = (\dot{p}_r - p_\theta^* \dot{\theta}) \hat{e}_r + (p_r \dot{\theta} + \dot{p}_\theta^*) \hat{e}_\theta + \dot{p}_z \hat{e}_z$$

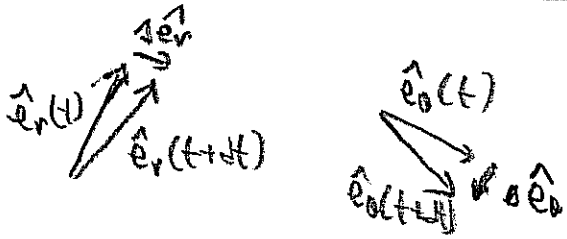
WHERE WE HAVE USED:

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} \qquad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$$

$$\Rightarrow \frac{dp}{dt} = \left[\frac{d}{dt}(\gamma m \dot{r}) - (\gamma m r \dot{\theta}^2)\right] \hat{e}_r$$
$$+ \underbrace{\left[\gamma m \dot{r} \dot{\theta} + \frac{d}{dt}(\gamma m r \dot{\theta})\right]}_{= \frac{1}{r}\frac{d}{dt}(\underbrace{\gamma m r^2 \dot{\theta}}_{\text{MECHANICAL ANGULAR MOMENTUM}})} \hat{e}_\theta + \frac{d}{dt}(\gamma m \dot{z}) \hat{e}_z$$

PROJECTION OF PARTICLE POSITION AT TIMES t & $t+dt$ ON X-Y PLANE





$$\Delta\hat{e}_r = \hat{e}_\theta \Delta\theta \qquad \Delta\hat{e}_\theta = -\hat{e}_r \Delta\theta$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\, \dot{\theta} \qquad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\, \dot{\theta}$$

CONSERVATION OF CANONICAL ANGULAR MOMENTUM

J. BAHNMAN

DEFINE $p_\theta = \gamma m r^2 \dot{\theta} + q r A_\theta$

$$\frac{d}{dt} p_\theta = 0$$ (CONSERVATION OF CANONICAL ANGULAR MOMENTUM)

NOTE THAT THE FLUX ENCLOSED BY A CIRCLE OF RADIUS r

$$\psi = \int \underline{B} \cdot d\underline{A} = \int (\nabla \times \underline{A}) \cdot dA = \oint \underline{A} \cdot d\underline{l} = 2\pi r A_\theta$$

$$p_\theta = \gamma m r^2 \dot{\theta} + \frac{q}{2\pi} \psi$$

IS CONSERVED ALONG AN ORBIT IN AXISYMMETRIC GEOMETRIES

"EXTERNAL" ELECTRIC & MAGNETIC FIELD WITH AZIMUTHAL SYMMETRY ($\frac{\partial}{\partial\theta} = 0$) (REISER SECTION 5.3)

CONSIDER THE FIELD $\underline{E}_{ext}$ & $\underline{B}_{ext}$ CREATED BY EXTERNAL SOURCES:

(TIME STEADY, VACUUM FIELDS):

$$\nabla \times \underline{B}_{ext} = 0 \qquad \underline{\nabla} \times \underline{E}_{ext} = 0 \qquad (\Rightarrow E, B \sim \nabla\phi)$$

$$\nabla \cdot \underline{B}_{ext} = 0 \qquad \underline{\nabla} \cdot \underline{E}_{ext} = 0 \qquad \Rightarrow \nabla^2\phi = 0$$

In cylindrical coordinates:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

Let $\Phi(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z)\, r^{2\nu} = f_0 + f_2 r^2 + f_4 r^4 + \ldots$

$$\nabla^2\Phi = 0 \quad \Rightarrow \quad \sum_{\nu=1}^{\infty} (2\nu)^2 f_{2\nu} r^{2\nu-2} + \sum_{\nu=0}^{\infty} f_{2\nu}'' r^{2\nu} = 0$$

$$\Rightarrow \quad \begin{aligned} f_0''(z) &= -4 f_2 \\ f_2''(z) &= -16 f_4 \\ f_4''(z) &= -36 f_6 \\ &\cdots \end{aligned} \quad \Rightarrow \quad f_{2\nu} = \frac{(-1)^\nu}{(\nu!)^2 2^{2\nu}} \frac{\partial^{2\nu} f(z)}{\partial z^{2\nu}}$$

$$\Rightarrow \Phi(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} f(z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu} = f_0(z) - \frac{1}{4} f_0''(z) r^2 + \frac{1}{64} f_0''''(z) r^4 + \ldots$$

Let $B_z(0,z) = B(z)$ & LET $\Phi(0,z) = V(z) = f_0(z)$

$$B_z(r,z) = \frac{-\partial\phi(r,z)}{\partial z} = f_0'(z) - \frac{1}{4} f_0'''(z) r^2 + \frac{1}{64} f_0'''''(z) r^4 + \ldots$$

$$= B(z) - \frac{r^2}{4}\frac{\partial^2 B(z)}{\partial z^2} + \frac{r^4}{64}\frac{\partial^4 B(z)}{\partial z^4} + \ldots$$

$$B_r(r,z) = \frac{-\partial\phi(r,z)}{\partial z} = -\frac{r}{2}\frac{\partial B(z)}{\partial z} + \frac{r^3}{16}\frac{\partial^3 B}{\partial z^3} + \ldots$$

SIMILARLY, for the electric field define

$$V(z) = \Phi(0,z) = f_0(z)$$

$$\Rightarrow \Phi(r,z) = V(z) - \frac{r^2}{4}V''(z) + \frac{r^4}{64}\frac{\partial^4 V}{\partial z^4} + \cdots$$

$$\Rightarrow E_r(r,z) = \frac{r}{2}V''(z) - \frac{r^3}{16}\frac{\partial^4 V(z)}{\partial z^4} + \cdots$$

$$\& \quad E_z(r,z) = -V'(z) + \frac{r^2}{4}V'''(z) + \cdots$$

RETURNING TO THE RADIAL COMPONENT OF THE MOMENTUM EQUATION IN CYLINDRICAL COORDINATES (EQ I):

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = q(E_r + r\dot{\theta} B_z - \dot{z} B_\theta) \qquad (I)$$

for the external field use

$$E_{r\,ext} = \frac{r}{2} V'' + O(r^3)$$

$$B_{z\,ext} = B_z(z) + O(r^3)$$

$$B_{\theta\,ext} = 0 \qquad \left[\text{since } \frac{\partial \phi_{mag}}{\partial \theta} = 0\right]$$

WE LET:

$$\underline{B} = \underline{B}_{ext} + \underline{B}_{self}$$

$$\underline{E} = \underline{E}_{ext} + \underline{E}_{self}$$

PARAXIAL RAY EQUATION

$$\text{(I)} \Rightarrow \quad \frac{d}{dt}(\gamma m \dot r) - \gamma m r \dot\theta^2 = q\left(\frac{V''}{2} r + r\dot\theta B\right) + q\left(E_r^{self} - v_z B_\theta^{self}\right)$$

$$\underbrace{\frac{d}{dt}(\gamma m \dot r)}_{\text{INERTIAL}} \quad \underbrace{\gamma m r \dot\theta^2}_{\text{CENTRIFUGAL}} \quad \underbrace{\frac{V''}{2} r}_{E_r \text{ external}} \quad \underbrace{r\dot\theta B}_{v_\theta B_z \text{ external}} \quad \underbrace{E_r^{self} - v_z B_\theta^{self}}_{\text{SELF FIELDS}}$$

Now use s as independent variable $\quad v_z\, dt = ds$

$$v_z \frac{d}{ds}(\gamma m v_z r') - \gamma m v_z^2 r \theta'^2 = q\left(\frac{V''}{2} r + r v_z \theta' B\right) + q\left(E_r^{self} - v_z B_\theta^{self}\right)$$

EXPANDING 1^{st} term and $v_z \cong v$; AND DIVIDING BY $\gamma m v^2$:

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r + r\beta c\,\theta' B + E_r^{self} - v_z B_\theta^{self}\right) \qquad \text{(P1)}$$

Using CANONICAL MOMENTUM, eliminate θ' via

$$\theta' = \frac{p_\theta - \frac{q\psi}{2\pi}}{\gamma m r^2 \beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{qB}{2\gamma\beta m c} \equiv \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$

where we define $\quad \omega_c \equiv \frac{qB}{m}$

ADDING THE TWO θ' TERMS IN THE EQUATION (P1)

$$-r\theta'^2 - \frac{r\omega_c\theta'}{\gamma\beta c} = \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} - \frac{r\omega_c^2}{4\gamma^2\beta^2 c^2}$$
$$\qquad - \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} + \frac{r\omega_c^2}{2\gamma^2\beta^2 c^2}$$
$$= \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{r\omega_c^2}{2\gamma^2\beta^2 c^2}$$

So equation (P1) becomes:

$$r'' + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r\right) + \frac{r\omega_c^2}{2\gamma^2\beta^2 c^2} + \frac{p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{q}{\gamma m \beta^2 c^2}\left[E_r^{self} - v_z B_\theta^{self}\right] \quad \text{(P2)}$$

Now $\gamma' mc^2 = q\frac{\underline{E}\cdot\underline{v}}{v_z}$ $\Rightarrow$ so $\gamma'' = -\left(V'' + \frac{\partial^2\phi^{self}}{\partial z^2}\right)\frac{q}{mc^2}$

<u>CALCULATING $\frac{q}{\gamma m \beta^2 c^2}\left[\frac{V''}{2} r + E_r^{self} - v_z B_\theta^{self}\right]$:</u>

$$\nabla^2 \phi^{self} = -\frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{\rho}{\varepsilon_0} - \frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{r\rho(r)}{\varepsilon_0} - \frac{r\,\partial^2\phi^{self}}{\partial z^2}$$

$$r\frac{\partial\phi}{\partial r} = -\frac{1}{2\pi\varepsilon_0}\int_0^r 2\pi r\,\rho(r)\,dr - \frac{r^2}{2}\frac{\partial^2\phi}{\partial z^2}$$

$$= -\frac{1}{2\pi\varepsilon_0}\lambda(r) - \frac{r^2}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\Rightarrow E_r^{self} = \frac{\lambda(r)}{2\pi\varepsilon_0 r} + \frac{r}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\nabla\times\underline{B} = \mu_0\underline{J} \quad \Rightarrow \quad 2\pi r B_\theta = \mu_0\int_0^r 2\pi r\, J_z(r)\,dr = \mu_0 v_z\lambda(r)$$

$$B_\theta^{self} = \frac{\mu_0 v_z\lambda(r)}{2\pi r} = \frac{v_z}{c^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

$$\left[\frac{V''}{2} r + E_r^{self} - v_z B_\theta^{self}\right] = \left[\frac{r}{2}\underbrace{\left(V'' + \frac{\partial^2\phi^{self}}{\partial z^2}\right)}_{-\frac{mc^2}{q}\gamma''} + \underbrace{\left(1 - \frac{v_z^2}{c^2}\right)}_{1/\gamma^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}\right]$$

So equation (P2) becomes: "THE PARAXIAL RAY EQUATION:"

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{\substack{E_r \\ \text{(CONVERGENCE OF FIELD LINES)}}} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{\substack{V_\theta B_z \\ -\text{ CENTRIFUGAL}}} - \underbrace{\left(\frac{P_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF FIELD}} = 0$$

J. BARNARD

MOMENT EQUATIONS

Vlasov eqtn: $\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} = 0$

Let $g = g(x, x', y, y')$; $N = \iiiint f \, dx\, dx'\, dy\, dy'$

MULTIPLY VLASOV equation by g & $\frac{1}{N}\iiiint dx\, dx'\, dy\, dy'$

$$\int dx\, dx'\, dy\, dy \left[g\frac{\partial f}{\partial s} + g x'\frac{\partial f}{\partial x} + g x''\frac{\partial f}{\partial x'} + g y'\frac{\partial f}{\partial y} + g y''\frac{\partial f}{\partial y'} \right] = 0$$

INTEGRATE BY PARTS

$$\Rightarrow \frac{d}{ds}\langle g\rangle + \underbrace{\frac{1}{N}\iiint g f \Big|_{x=-\infty}^{\infty}}_{\to 0} - \underbrace{\frac{1}{N}\iiiint \frac{\partial g}{\partial x} f x'}_{= \langle x' \frac{\partial g}{\partial x}\rangle} + \cdots = 0$$

$$\Rightarrow \frac{d}{ds}\langle g\rangle = \langle x'\frac{\partial g}{\partial x}\rangle + \langle x''\frac{\partial g}{\partial x'}\rangle + \langle y'\frac{\partial g}{\partial y}\rangle + \langle y''\frac{\partial g}{\partial y'}\rangle$$

But $\frac{dg}{ds} = \frac{\partial g}{\partial x}x' + \frac{\partial g}{\partial x'}x'' + \frac{\partial g}{\partial y}y' + \frac{\partial g}{\partial y'}y''$

$$\Rightarrow \frac{d}{ds}\langle g\rangle = \langle g'\rangle$$

So $\frac{d}{ds}\langle x^2\rangle = 2\langle x x'\rangle$

$\frac{d}{ds}\langle x'^2\rangle = 2\langle x' x''\rangle$ etc...

$\frac{d}{ds}\langle x x'\rangle = \langle x x''\rangle + \langle x'^2\rangle$

ENVELOPE EQUATION FOR AXISYMMETRIC BEAMS

LET $r_b^2 = 2\langle r^2\rangle = 2(\langle x^2\rangle + \langle y^2\rangle) = 4\langle x^2\rangle$ for an axisymmetric beam

$2 r_b r_b' = 4\langle r r'\rangle \quad \Rightarrow \quad r_b' = \dfrac{2\langle r r'\rangle}{r_b}$

$$r_b'' = \frac{2\langle r r''\rangle + 2\langle r'^2\rangle}{r_b} - \frac{2\langle r r'\rangle}{r_b^2}\left(\frac{2\langle r r'\rangle}{r_b}\right)$$

$$= 2\,\frac{\langle r r''\rangle}{r_b} + \frac{4\langle r^2\rangle\langle r'^2\rangle - 4\langle r r'\rangle^2}{r_b^3}$$

WHAT IS $\langle r r''\rangle$?

RECALL EQUATION P1 (ON PATH TO PARAXIAL RAY EQUATION):

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r + r\beta c\theta' B + E_r^{self} - v_z B_z^{self}\right)$$

P1 may be rewritten:

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left[\frac{-mc^2}{q}\gamma''\frac{r}{2} + \frac{\lambda(r)}{\gamma^2 2\pi\epsilon_0 r} + r\beta c\theta' B\right]$$

$$\Rightarrow$$

$$\boxed{r'' + \frac{\gamma'}{\beta^2\gamma} r' + \frac{\gamma''}{2\beta^2\gamma} r - \frac{q}{\gamma^3 m v_z^2}\frac{\lambda(r)}{2\pi\epsilon_0 r} - \frac{\omega_c}{\gamma\beta c}\theta' r - r\theta'^2 = 0}$$

What is $\langle r r''\rangle$?

$$\langle r r''\rangle + \frac{-\omega_c}{\gamma\beta c}\langle\theta' r^2\rangle - \langle r^2\theta'^2\rangle + \dots = 0$$

$$\langle p_\theta\rangle^2 = \gamma^2 m^2 \beta^2 c^2 \langle r^2\theta'\rangle^2 + \frac{\omega_c^2}{4} m^2\langle r^2\rangle^2 + \omega_c \gamma m^2 \beta c \langle r^2\theta'\rangle\langle r^2\rangle$$

$$\Rightarrow \frac{-\omega_c}{\gamma\beta c}\langle\theta' r^2\rangle = \frac{-\omega_c}{\gamma\beta c}\left[\frac{\langle p_\theta\rangle^2}{\omega_c\gamma m^2\beta c\langle r^2\rangle} - \frac{\omega_c\langle r^2\rangle}{4\gamma\beta c} - \frac{\gamma\beta c}{\omega_c}\frac{\langle r^2\theta'\rangle^2}{\langle r^2\rangle}\right]$$

$$\Rightarrow \langle r r''\rangle - \frac{\langle p_\theta\rangle^2}{\gamma^2 m^2\beta^2 c^2\langle r^2\rangle} + \frac{\omega_c^2\langle r^2\rangle}{4\gamma^2\beta^2 c^2} + \frac{\langle r^2\theta'\rangle^2}{\langle r^2\rangle} - \langle r^2\theta'^2\rangle + \dots = 0$$

$\Rightarrow$

$$\langle r r'' \rangle = \frac{\gamma'}{\beta^2 \gamma} \langle r r' \rangle + \frac{\gamma''}{2\beta^2\gamma} \langle r^2 \rangle - \frac{q}{\gamma^3 m v_z^2} \frac{\langle \lambda(r) \rangle}{2\pi \epsilon_0} +$$

$$\frac{\langle p_\theta \rangle^2}{(\gamma m \beta c)^2 \langle r^2 \rangle} - \frac{\omega_c^2 \langle r^2 \rangle}{4(\gamma \beta c)^2} - \frac{\langle r^2 \theta' \rangle^2}{\langle r^2 \rangle} + \langle r^2 \theta'^2 \rangle$$

$$r_b'' = \frac{2\langle r r'' \rangle}{r_b} + \frac{4\langle r^2 \rangle \langle r'^2 \rangle - 4 \langle r r' \rangle}{r_b^3}$$

$$= \frac{\gamma'}{\beta^2\gamma} \frac{2\langle r r' \rangle}{r_b} + \frac{\gamma''}{2\beta^2\gamma} \frac{2\langle r^2 \rangle}{r_b} - \frac{2q}{\gamma^3 m v_z^2} \frac{\langle \lambda(r) \rangle}{2\pi\epsilon_0} \frac{1}{r_b}$$

$$+ \frac{\langle p_\theta \rangle^2}{(\gamma m \beta c)^2} \frac{2}{\langle r^2 \rangle r_b} - \frac{\omega_c^2}{4(\gamma\beta c)^2} \frac{2\langle r^2 \rangle}{r_b} - \frac{2\langle r^2 \theta' \rangle^2}{r_b \langle r^2 \rangle}$$

$$+ \frac{2\langle r^2 \theta'^2 \rangle}{r_b} + \frac{4\langle r^2 \rangle \langle r'^2 \rangle - 4\langle r r' \rangle^2}{r_b^3}$$

USING $r_b^2 \equiv 2\langle r^2 \rangle$ & $r_b' = \dfrac{2\langle r r' \rangle}{r_b}$

ENVELOPE EQUATION

$$\Rightarrow \boxed{r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b + \frac{-4\langle p_\theta \rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0}$$

WHERE $\varepsilon_r^2 = 4\left(\langle r^2 \rangle \langle r'^2 \rangle - \langle r r' \rangle^2 + \langle r^2 \rangle \langle r^2 \theta'^2 \rangle - \langle r^2 \theta' \rangle^2\right)$

ENVELOPE EQUATION -- CONTINUED

$$\boxed{r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0}$$

COMPARE WITH THE SINGLE PARTICLE PARAXIAL RAY EQUATION:

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{E_r} + \underbrace{\left(\frac{\omega_L}{2\gamma\beta c}\right)^2 r}_{v_\theta B_z - \text{CENTRIFUGAL}} - \underbrace{\left(\frac{p_\theta}{\gamma m\beta c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}}_{E_r - v_z B_\theta \text{ self field}} = 0$$

$$\varepsilon_r^2 = 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle rr'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$

NOTE THAT FOR AXISYMMETRIC BEAMS ($\rho = \rho(r)$ ONLY)

$$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle = 2\langle x^2\rangle$$

$$\Rightarrow 2\langle rr'\rangle = 4\langle xx'\rangle$$

$$\& \quad \langle x'^2\rangle + \langle y'^2\rangle = 2\langle x'^2\rangle = \langle r'^2\rangle + \langle r^2\theta'^2\rangle$$

DEFINE $\varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)$

$$\Rightarrow \boxed{\varepsilon_r^2 = \varepsilon_x^2 - 4\langle r^2\theta'\rangle^2}$$

EXAMPLES OF
SYSTEMS WITH AXIAL SYMMETRY

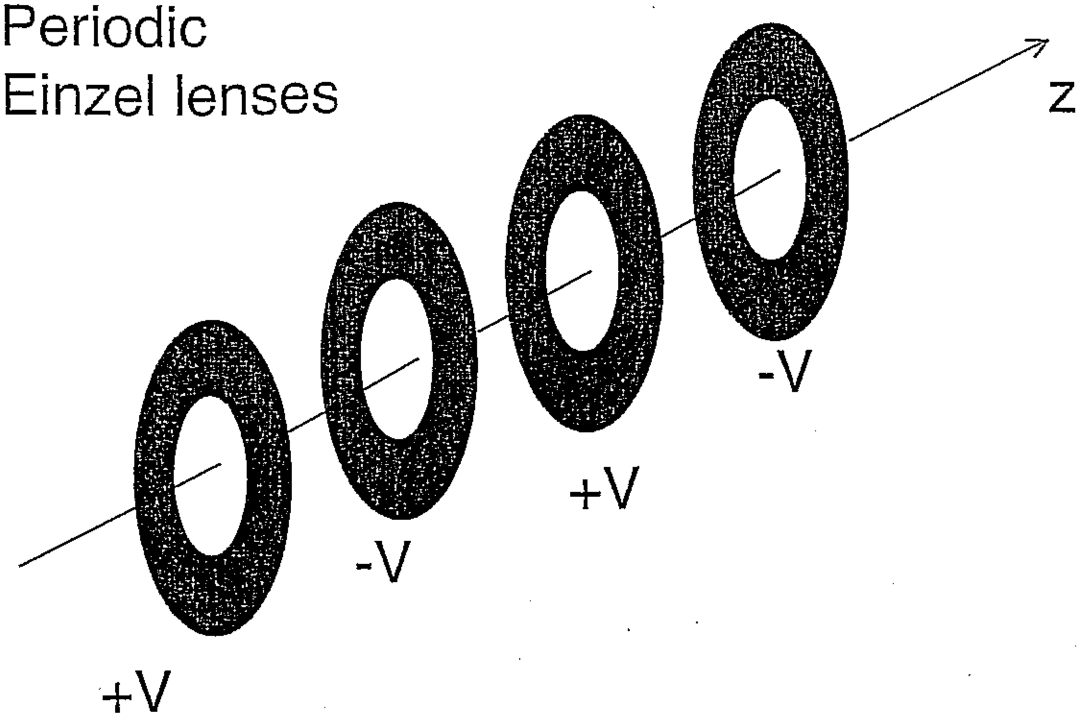
- PERIODIC SOLENOIDS
- EINZEL LENSES
- CONTINUOUS FOCUSING

EXAMPLES OF
SYSTEMS WITHOUT AXIAL SYMMETRY

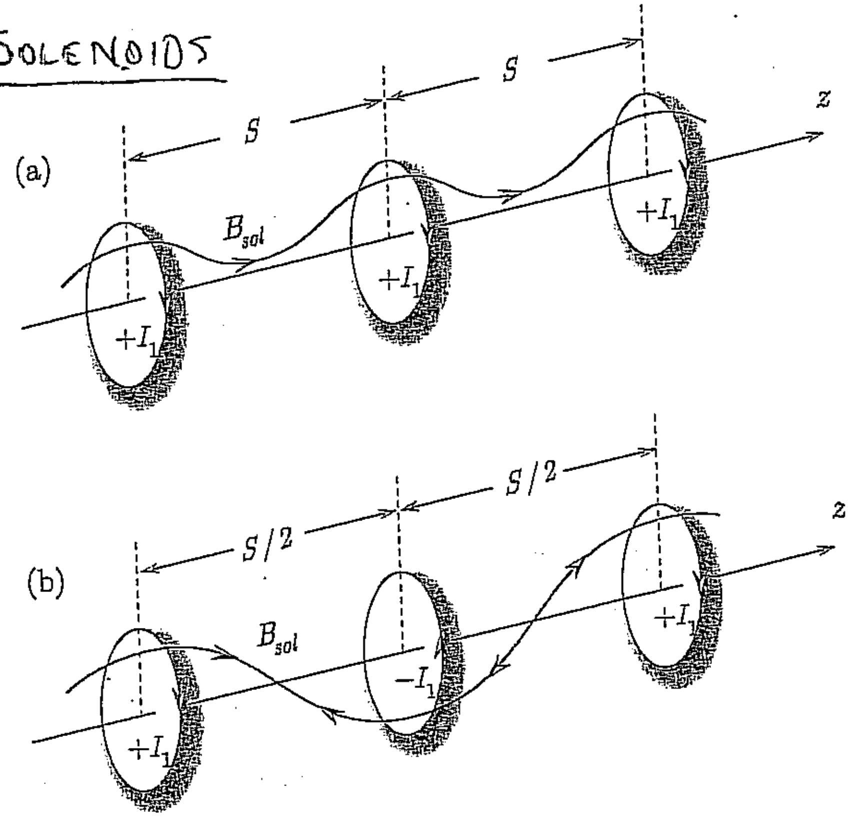
- ELECTRIC OR MAGNETIC QUADRUPOLE

$\Rightarrow$ USE CARTESIAN COORDINATES WITH
ELLIPTICAL SPACE CHARGE SYMMETRY

EXAMPLES OF AXISYMMETRIC SYSTEMS



PERIODIC SOLENOIDS



(FIGURE FROM DAVIDSON & QIN 2003) p. 55 "PHYSICS OF INTENSE CHARGE PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S. In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \cdots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \cdots$.

EXAMPLE OF NON-AXISYMMETRIC SYSTEM

(20)

figure from
Davidson & Qin, 2003.

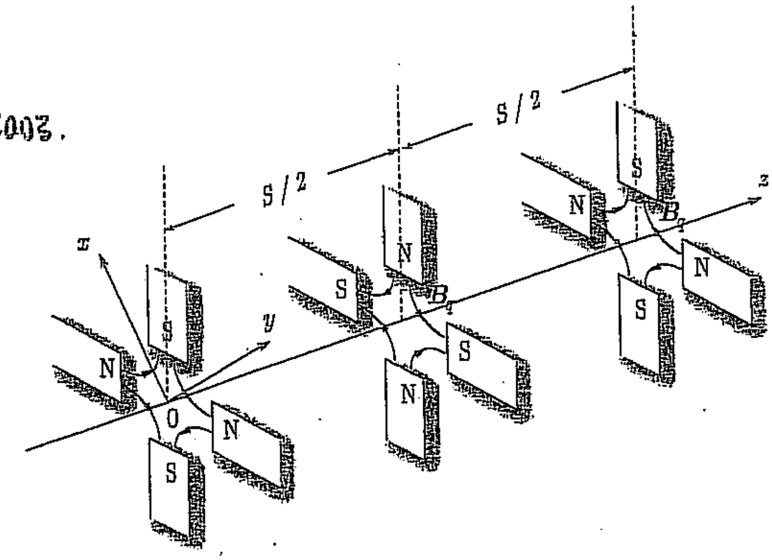


Figure 3.1. Schematic of magnet sets producing an alternating-gradient quadrupole field with axial periodicity length S.

NON-AXISYMMETRIC SYSTEM

(21)

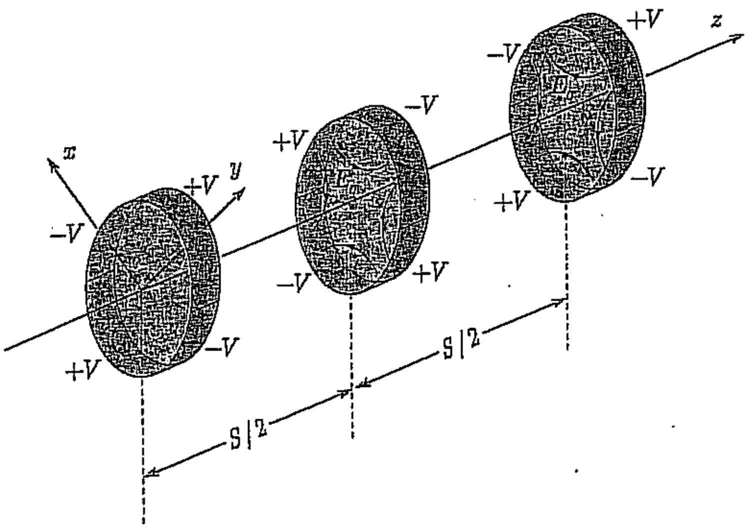


Figure 3.3. Schematic of conductor configuration with applied voltages producing an alternating-gradient quadrupole electric field with axial periodicity length S.

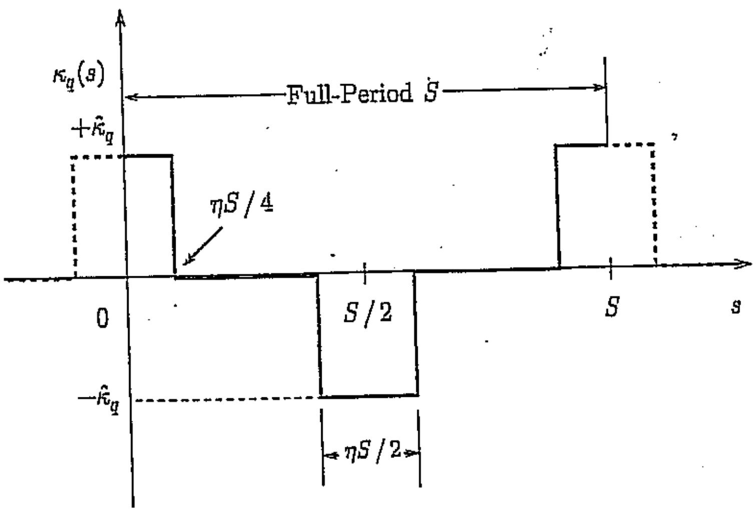


Figure 3.7. Alternating step-function model of a periodic quadrupole lattice with filling factor η for the lens elements. The figure shows a plot of the quadrupole coupling coefficient $\kappa_q(s)$ versus s for one full period (S) of the lattice. Such a configuration is often called a FODO transport lattice (acronym for focusing-off-defocusing-off).

FIGURES FROM DAVIDSON & QIN 2003

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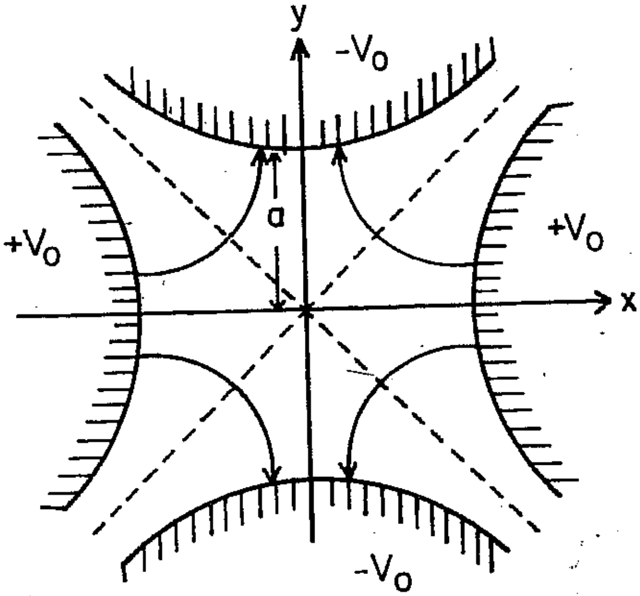
(22)

From Reiser, p. 112

$E_x = -E'x$

$E_y = E'y$



$F_x = -qE'x$

$F_y = qE'y$

Electrostatic Quads

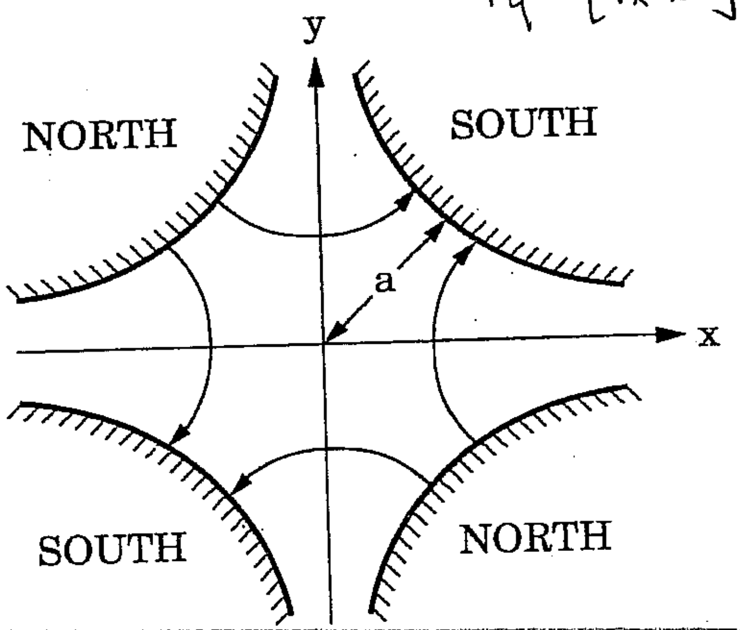
Figure 3.15. Electrodes and force lines in an electrostatic quadrupole.

$B_x = B'y$

$B_y = B'x$

$F_x = -qV_z B'x$

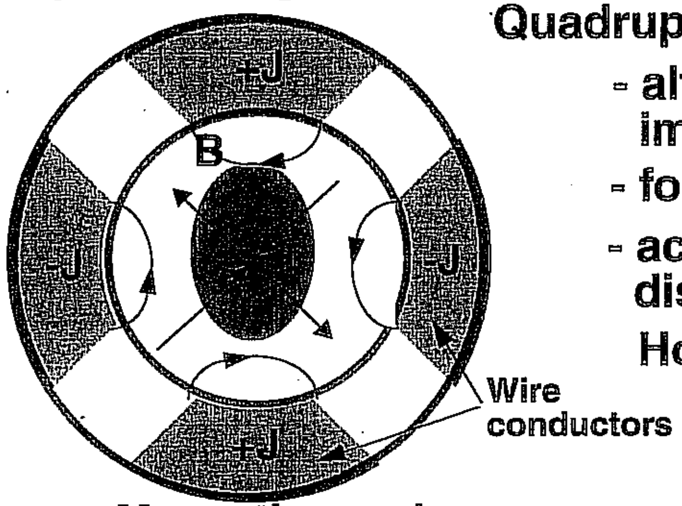
$F_y = qV_x B'y$



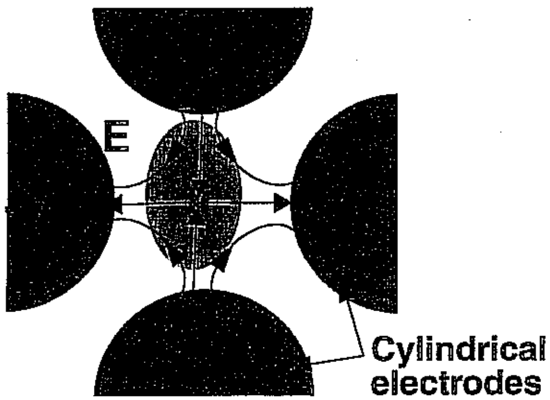
Magnetic Quads

Heavy ion accelerators use alternating gradient quadrupoles to focus (confine) the beams (non-neutral plasmas)

Space-charge forces and thermal forces act to expand beam



Magnetic quad

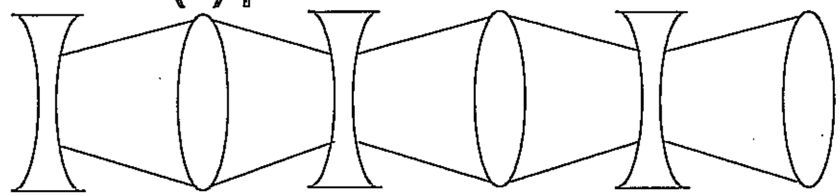


Electric quad

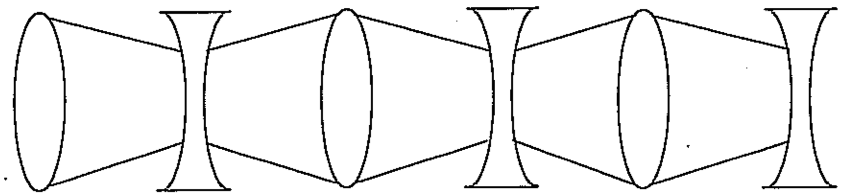
Quadrupoles (magnetic or electric):

- alternately provide inward then outward impulse
- focus in one plane and defocus in other
- act as linear lenses. (Force proportional to distance from axis).

Horizontal (x) plane:



Vertical (y) plane:



Average displacement
is larger in focusing lenses
so the net effect is focusing.

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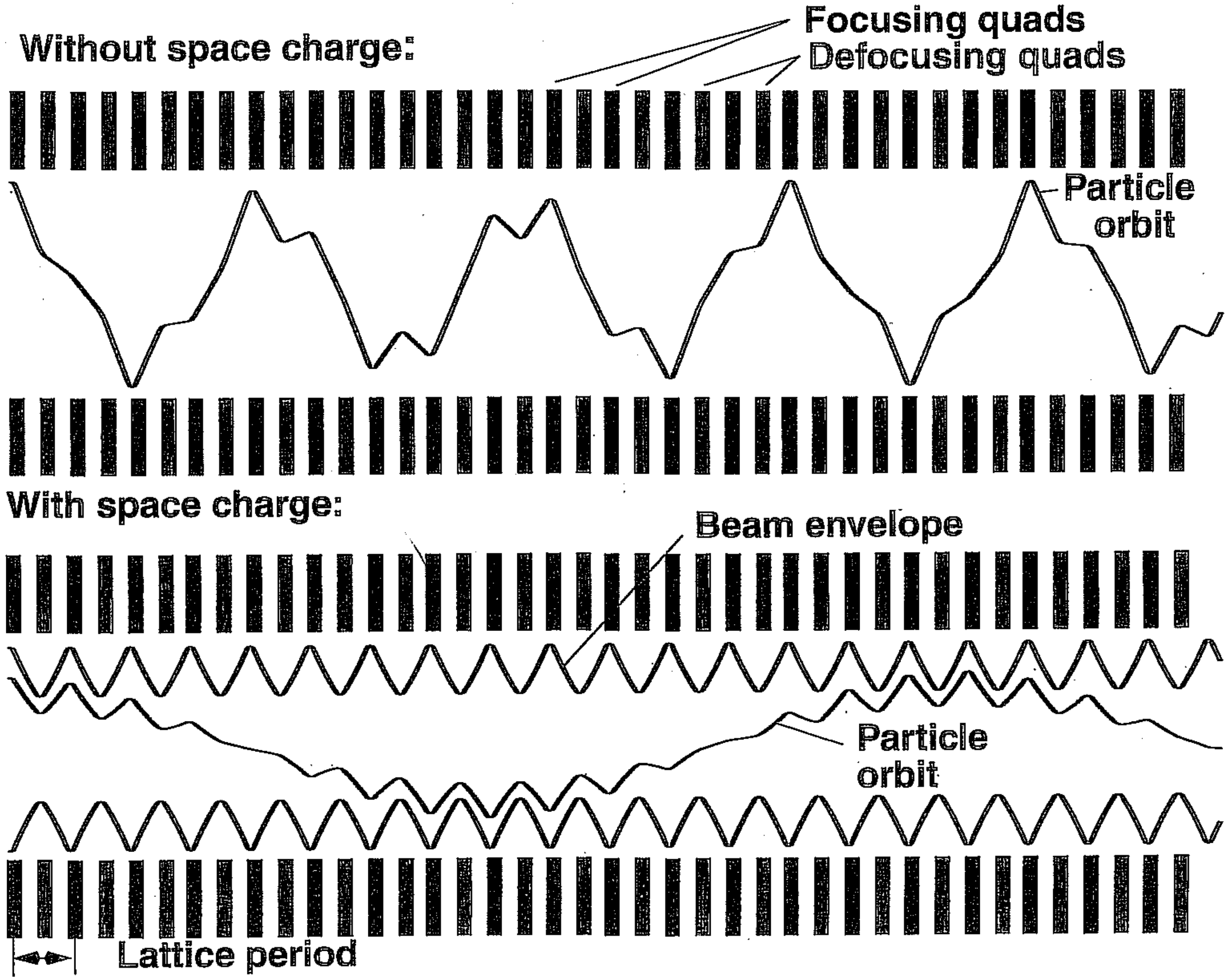
23

Space charge reduces betatron phase advance



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(24)

ENVELOPE EQUATIONS FOR NON-AXISYMMETRIC SYSTEMS

$r_x^2 \equiv 4\langle x^2 \rangle \qquad r_y^2 \equiv 4\langle y^2 \rangle$

$2 r_x r_x' = 8\langle x x' \rangle$

$$r_x' = \frac{4\langle x x' \rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x'' \rangle}{r_x} + \frac{4\langle x'^2 \rangle}{r_x} - \frac{4\langle x x' \rangle}{r_x^2} r_x'$$

$$= \frac{4\langle x x'' \rangle}{r_x} + \frac{16\langle x'^2 \rangle\langle x^2 \rangle}{r_x^3} - \frac{16\langle x x' \rangle^2}{r_x^3}$$

DEFINE $\varepsilon_x^2 = 16\left(\langle x^2 \rangle\langle x'^2 \rangle - \langle x x' \rangle^2\right)$

$$\Rightarrow \boxed{r_x'' = \frac{4\langle x x'' \rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}}$$

SO HOW DO WE CALCULATE $\langle x x'' \rangle$?

RETURN TO SINGLE PARTICLE EQUATION (IN CARTESIAN COORDINATES)

$$\frac{d}{dt}(\gamma m \dot{x}) = \dot{\gamma} m \dot{x} + \gamma m \ddot{x} = q\left(E_x + \dot{y} B_z - \dot{z} B_y\right)$$

$\downarrow$ x'' & similarly y''

$\downarrow$ QUADRUPOLE FOCUSING, SPACE-CHARGE OF ELLIPTICAL BEAMS

TO BE CONTINUED . . .

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QUADRUPOLE FOCUSING

Now, relax radial symmetry:

For $\nabla\cdot E = 0$ or $\nabla\times B = 0$

EXPAND FIELD IN CYLINDRICAL "MULTIPOLES":

$$E_r, B_r = \sum_{n=1}^{\infty} f_n r^{n-1} \cos(n\theta)$$

$E_x = E_r\cos\theta - E_\theta \sin\theta$
$E_y = E_r \sin\theta + E_\theta\cos\theta$

$$E_\theta, B_\theta = \sum_{n=1}^{\infty} f_n r^{n-1} \sin(n\theta)$$

$n=1 \Rightarrow$ dipole $\begin{cases} E_r = f_1\cos\theta \\ E_\theta = -f_1\sin\theta \end{cases} \Rightarrow \begin{matrix} E_x = f_1 \\ E_y = 0 \end{matrix}$

$n=2 \Rightarrow$ quadrupole $\begin{cases} E_r = f_2 r\cos 2\theta \\ E_\theta = -f_2 r\sin 2\theta \end{cases} \Rightarrow \begin{matrix} E_x = f_2 x \\ E_y = -f_2 y \end{matrix}$

NOTE: ABOVE EXPANSION IS VALID WHEN E & $B \neq$ function(z).
FOR MAGNETS OF FINITE AXIAL EXTENT, FOR EACH FUNDAMENTAL n-pole, A SET OF HIGHER ORDER MULTIPOLES WITH SAME AZIMUTHAL SYMMETRY ARE REQUIRED TO SATISFY $\nabla^2\phi = 0$.

FOR EXAMPLE FOR A FUNDAMENTAL QUADRUPOLE THE FIELD MAY BE EXPANDED:

$$E_r = \sum_{\nu=0}^{\infty} f_{2,\nu}(z)\,[1+\nu]\, r^{1+2\nu} \cos[2\theta]$$

$$E_\theta = \sum_{\nu=0}^{\infty} -f_{2,\nu}(z)\, r^{1+2\nu} \sin[2\theta]$$

$$E_z = \sum_{\nu=0}^{\infty} \frac{1}{2}\frac{df_{2,\nu}}{dz} r^{2+2\nu} \cos 2\theta$$

with $f_{2,\nu+1}(z) = \dfrac{-1}{4(\nu+1)(\nu+3)} \dfrac{d^2 f_{2,\nu}(z)}{dz^2}$

SEE LUND, S. M. (1996) FOR EXAMPLE. HIF note 96-? LLNL.