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NE-290H
Spring 2009
U.C. Berkeley

Injectors and longitudinal physics -- I

1. Fluid equations
2. Child-Langmuir Law
   (Reiser 2.5.2, Appendix 1)
3. Pierce electrodes
4. Transients in injectors
5. Injector choices

I) FLUID EQUATIONS

START WITH VLASOV EQUATION FOR $f(\underline{x},\underline{p},t)$

$$\frac{\partial f}{\partial t}(\underline{x},\underline{p},t) + \dot{\underline{x}}\cdot\frac{\partial f}{\partial \underline{x}}(\underline{x},\underline{p},t) + \dot{\underline{p}}\cdot\frac{\partial f}{\partial \underline{p}}(\underline{x},\underline{p},t) = 0$$

HERE $\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m}$

$$\dot{\underline{p}} = \frac{d\underline{p}}{dt} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{\gamma m}\times\underline{B}(\underline{x},t)\right)$$

$$\gamma^2 = (\underline{p}/mc)^2 + 1$$

INTEGRATE OVER MOMENTUM AND MULTIPLY BY POWERS OF $\underline{p}$

a) CONTINUITY EQUATION

$$\int d^3p\left\{\overset{①}{\frac{\partial f}{\partial t}} + \overset{②}{\dot{\underline{x}}\cdot\frac{\partial f}{\partial \underline{x}}} + \overset{③}{\left(q\underline{E}(\underline{x},t) + \frac{q\underline{p}}{\gamma m}\times \underline{B}(\underline{x},t)\right)\frac{\partial f}{\partial \underline{p}}}\right\} = 0$$

DEFINE $n(\underline{x},t) = \int f(\underline{x},\underline{p},t)\,d^3p$

$$n(\underline{x},t)\,\underline{\bar{V}}(\underline{x},t) = \int \frac{\underline{p}}{\gamma m} f(\underline{x},\underline{p},t)\,d^3p$$

① FIRST INTEGRAL

$$\int d^3p\,\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int f\,d^3p = \frac{\partial n(\underline{x},t)}{\partial t}$$

② SECOND INTEGRAL

$$\int d^3p\,\dot{\underline{x}}\cdot\frac{\partial f}{\partial \underline{x}} = \int d^3p\,\frac{\underline{p}}{\gamma m}\cdot\frac{\partial f}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}}\cdot\int d^3p\,\frac{\underline{p}}{\gamma m} f\,d^3p$$

$$= \frac{\partial}{\partial \underline{x}}\cdot n\underline{\bar{V}}$$

(3) THIRD INTEGRAL

$$\int d^3p \left(q\underline{E} + \frac{q}{\gamma m} \underline{p} \times \underline{B} \right) \frac{\partial f}{\partial \underline{p}}$$

$$\underbrace{q E f \Big|_{p=-\infty}^{p=+\infty}}_{=0}$$

$$\int \frac{q}{\gamma m} (p_y B_z - p_z B_y) \frac{\partial f}{\partial p_x} \, dp_x dp_y dp_z + \cdots$$

$$\int_{-\infty}^{\infty} u v' \, dp_x = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u' v \, dp_x$$

$u = \frac{q}{\gamma m}(p_y B_z - p_z B_y)$

$u' = \frac{-q}{\gamma^2 m}(p_y B_z - p_z B_y) \frac{\partial \gamma}{\partial p_x}$

$v \;\; f$

$v' \;\; \frac{\partial f}{\partial p_x}$

$$\gamma^2 = \frac{p_x^2 + p_y^2 + p_z^2}{m^2 c^2} + 1$$

$$\Rightarrow 2\gamma \frac{\partial \gamma}{\partial p_x} = \frac{2 p_x}{m^2 c^2}$$

$$\int \frac{q}{\gamma^3 m^2 c^2} (p_y B_z - p_z B_y) p_x + \frac{q}{\gamma^3 m^3 c^2} (p_z B_x - p_x B_z) p_y + \frac{q}{\gamma^3 m^3 c^2} (p_x B_y - p_y B_x) p_z$$

$$= 0 \;\; !$$

So $$\int d^3p \left\{ \frac{\partial f}{\partial t} + \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}} + \dot{\underline{p}} \cdot \frac{\partial f}{\partial \underline{p}} \right\} = 0$$

$\Rightarrow$

$$\boxed{\frac{\partial n(\underline{x},t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x},t) \, \overline{v}(\underline{x},t) = 0}$$

CONTINUITY EQUATION

$q n(\underline{x},t) \, \overline{v}(x,t) = \underline{J}(x,t)$

CURRENT DENSITY

ALTERNATIVELY $\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$

& BARNARD & LUND

b) MOMENTUM EQUATION

(FOR SIMPLICITY: ASSUME NON-RELATIVISTIC)

$$\dot{\underline{x}} = \frac{\underline{p}}{m} \qquad \dot{\underline{p}} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{m} \times \underline{B}(\underline{x},t)\right)$$

MULTIPLY BY $\dot{\underline{x}}$ & INTEGRATE OVER MOMENTUM (d^3p)

$$\int d^3p \left\{ \overset{(1)}{\dot{\underline{x}} \frac{\partial f}{\partial t}} + \overset{(2)}{\dot{\underline{x}}\, \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}}} + \overset{(3)}{\dot{\underline{x}} \left(q\underline{E} + \frac{\underline{p}}{m} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{p}}} \right\} = 0$$

DEFINE $\underline{\underline{P}} \equiv m \int d^3p\, (\dot{\underline{x}} - \overline{\underline{V}})(\dot{\underline{x}} - \overline{\underline{V}})\, f(x,p,t)$

($\underline{\underline{P}}$ = pressure tensor)

$$= m \int d^3p\, \dot{\underline{x}}\dot{\underline{x}} f - 2m\overline{\underline{V}} \int \dot{\underline{x}} f\, d^3p + m\overline{\underline{V}}\,\overline{\underline{V}} \int f d^3p$$

$$= m \int d^3p\, \dot{\underline{x}}\dot{\underline{x}} f - mn\overline{\underline{V}}\,\overline{\underline{V}}$$

(1) FIRST INTEGRAL:

$$\int d^3p\, \dot{\underline{x}} \frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d^3p\, \dot{\underline{x}} f = \frac{\partial}{\partial t} n\overline{\underline{V}}$$

(2) SECOND INTEGRAL:

$$\int d^3p\, \dot{\underline{x}}\, \dot{\underline{x}} \cdot \frac{\partial f}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}} \cdot \int d^3p\, \dot{\underline{x}}\, \dot{\underline{x}} f$$

$$= \frac{1}{m} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}} + \frac{\partial}{\partial \underline{x}} \cdot n\overline{\underline{V}}\,\overline{\underline{V}}$$

$$= \frac{1}{m} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}} + \left(\frac{\partial}{\partial \underline{x}} \cdot n\underline{V}\right)\underline{V} + n\underline{V} \cdot \frac{\partial \underline{V}}{\partial \underline{x}}$$

(3) THIRD INTEGRAL:

$$\int d^3p\, \frac{\underline{p}}{m} \left(q\underline{E} + q\frac{\underline{p}}{m} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{p}}$$

$u \quad \frac{p_x}{m}\left(q\underline{E} + \frac{\underline{p}}{m} \times B\right)$

$u' \quad \frac{1}{m}\left(q\underline{E} + \frac{\underline{p}}{m} \times B\right)$

$v \quad f$

$v' \quad \frac{\partial f}{\partial p_x}$

$$= \underbrace{f\, \frac{p_x}{m}\left(qE + \frac{q\underline{p}}{m} \times B\right)\Big|_{-\infty}^{\infty}}_{=0} - \int d^3p\, \frac{\underline{p}}{m}\left(q\underline{E} + \frac{q\underline{p}}{m} \times \underline{B}\right) f$$

$$= -\frac{n(\underline{x},t)}{m}\left(q\underline{E} + q\overline{\underline{V}} \times \underline{B}\right)$$

ADDING THE INTEGRALS TOGETHER:

$$\frac{\partial}{\partial t} n\bar{v} + \left(\frac{\partial}{\partial \underline{x}} \cdot n\underline{\bar{v}}\right)\underline{\bar{v}} + n\underline{\bar{v}} \cdot \frac{\partial \bar{v}}{\partial \underline{x}} = n \frac{q}{m}(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \frac{1}{m}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}$$

$$n\frac{\partial \bar{v}}{\partial t} + \underbrace{\frac{\partial n}{\partial t}\underline{\bar{v}} + \left(\frac{\partial}{\partial x} \cdot n\underline{\bar{v}}\right)\underline{\bar{v}}}_{\substack{=0 \\ \text{BY CONTINUITY} \\ \text{EQUATION}}} + n\underline{\bar{v}} \cdot \frac{\partial \bar{v}}{\partial \underline{x}} = \quad '' \qquad ''$$

$\rho(\underline{x},t) = m\, n(\underline{x},t)$

DIVIDING BY n:

$$\boxed{\frac{\partial \bar{v}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \bar{v}}{\partial \underline{x}} = \frac{q}{m}(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \frac{1}{\rho}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}}$$

(NON-RELATIVISTIC)

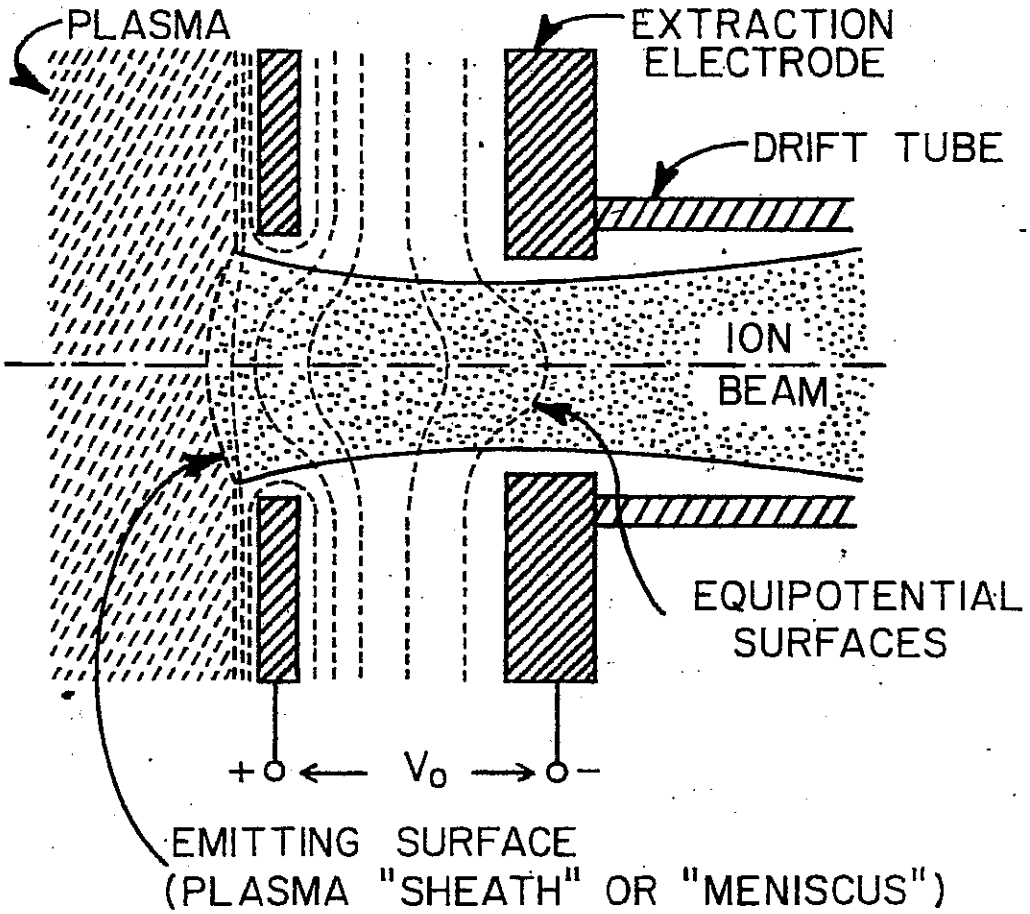
MOMENTUM EQUATION

NOTE THAT $\frac{\partial \bar{v}}{\partial t} + \underline{\bar{v}} \cdot \frac{\partial \bar{v}}{\partial \underline{x}} = \left.\frac{d}{dt}\underline{\bar{v}}\right|_{\text{ALONG A TRAJECTORY}}$

$$\Rightarrow \boxed{\frac{d\bar{v}}{dt} = \frac{q}{m}(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \frac{1}{\rho}\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}}$$

(NON-RELATIVISTIC)

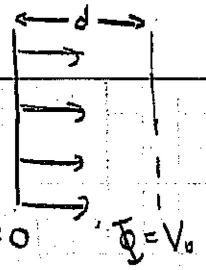
INJECTORS



OR "HOT PLATE"

REISER, FIGURE 1.2

- DOPED TUNGSTEN
- ALUMINO-SILICATE



I. CHILD-LANGMUIR EMISSION

ASSUME EMISSION IS PLANAR 1-D:

$$J = \rho v_z \qquad (1)$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = q \frac{\partial \Phi}{\partial z} \Rightarrow \frac{1}{2} m v_z^2 = q\,\Phi(z) \qquad (2) \quad \Rightarrow \quad v_z = \left(\frac{2q\Phi}{m}\right)^{1/2}$$

$$\frac{d^2\Phi}{dz^2} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

(NOTE $\Phi = -\phi$ ← ACTUAL e.s. potential)

CONTINUITY EQUATION (1D) $\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} \rho v_z = 0$

for time steady emission $\rho v_z = \text{constant} = J$

$$\Rightarrow \frac{d^2\Phi}{dz^2} = \frac{J}{\varepsilon_0 v_z} = \frac{J}{\varepsilon_0 \left(\frac{2q\Phi}{m}\right)^{1/2}}$$

MULTIPLYING BY $\frac{d\Phi}{dz}$ AND INTEGRATING:

$$\frac{\Phi'^2}{2} = \frac{J m^{1/2}}{\varepsilon_0 (2q)^{1/2}}\, 2\Phi^{1/2} + \text{const}$$

Assume $\Phi' = 0$ at $z = 0$ (Space-charge limited emission)

$\Phi = 0$ at $z=0 \Rightarrow$ const $= 0$

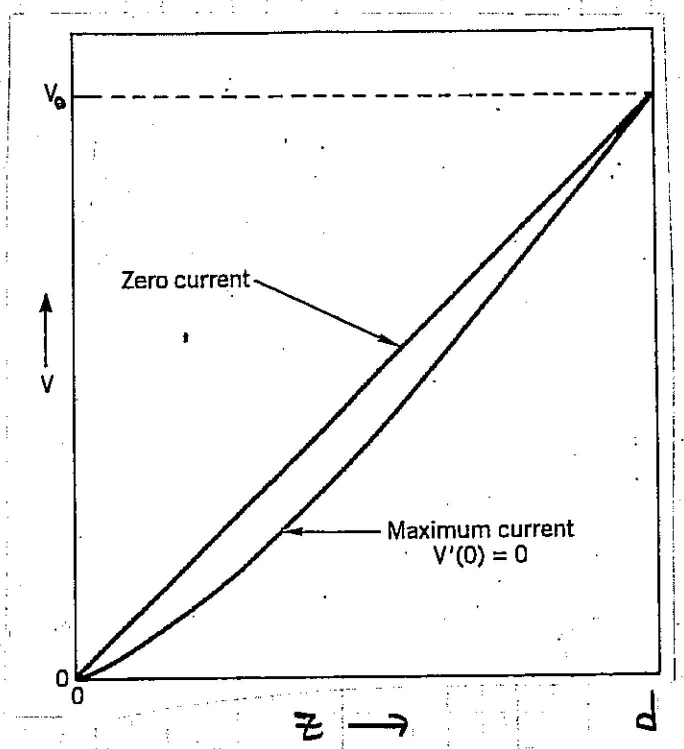
$$\frac{d\Phi}{dz} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2} \left(\frac{m\Phi}{2q}\right)^{1/4}$$

$$\Rightarrow \frac{4}{3}\Phi^{3/4} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2}\left(\frac{m}{2q}\right)^{1/4} z \quad \Rightarrow \Phi(x) = \left(\frac{3}{4}\right)^{4/3}\left(\frac{4J}{\varepsilon_0}\right)^{2/3}\left(\frac{m}{2q}\right)^{1/3} z^{4/3}$$

If $\Phi = V_0$ at $z = d$ $\Rightarrow$ $\Phi = V_0 \left(\frac{z}{d}\right)^{4/3}$

$$\Rightarrow \quad V_0 = \left(\frac{3}{4}\right)^{4/3} \left(\frac{4J}{\varepsilon_0}\right)^{2/3} \left(\frac{m}{2q}\right)^{1/3} d^{4/3}$$

$$\text{or} \quad J = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$



NOTE THAT IF WE MULTIPLY J BY THE BEAM AREA πr_b^2,
AND DIVIDE BY $v = \left(\frac{2qV_0}{m}\right)^{1/2}$

$$\Longrightarrow \lambda = \frac{4\pi\varepsilon_0 V}{9}\left(\frac{r_b^2}{d^2}\right) \qquad \text{RECALL } Q \equiv \frac{\lambda}{4\pi\varepsilon_0 V} \text{ (NON-REL'AT.)}$$

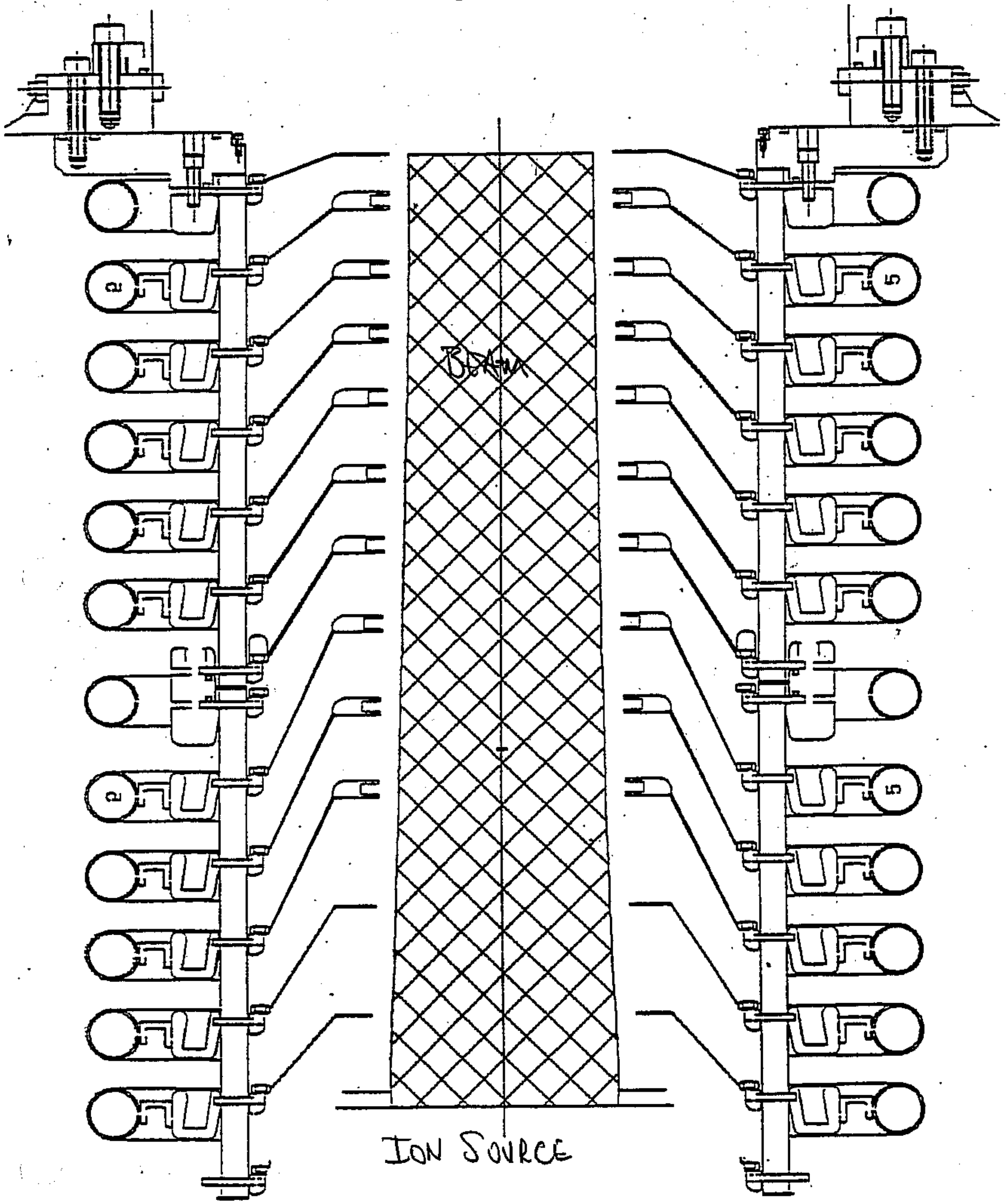
$$\Longrightarrow Q = \frac{1}{9}\left(\frac{r_b^2}{d^2}\right) \qquad \text{or as a function of } z\text{:} \quad Q(z) = \frac{1}{9}\left(\frac{r_b^2}{z^2}\right)$$

PERVEANCE

PIERCE COLUMN

$V \sim z^{4/3}$

$E \sim z^{1/3}$



WE DERIVED THE PARAXIAL RAY EQUATION FOR PARTICLES IN AXISYMMETRIC SYSTEMS:

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{\substack{v_\theta B_z \\ -\text{ CENTRIFUGAL}}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2}\,\frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\substack{\text{SELF} \\ \text{FIELD}}} = 0$$

$$\theta' = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$ ← CONSTANCY & DEFINITION OF CANONICAL MOMENTUM

ENVELOPE EQUATION FOR AXISYMMETRIC BEAM

$$r_b'' + \frac{\gamma' r_b'}{\beta^2\gamma} + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\varepsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$

RETURNING TO PARAXIAL ENVELOPE EQUATION:

$$(\text{for } \beta \ll 1) \quad r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

$$\text{for } r_b'' = \frac{\beta'}{\beta} r_b' = 0$$

$$\text{If} \quad \Phi = V_0 (z/d)^{4/3}$$

$$v = C z^{2/3}$$

$$v' = \frac{2}{3} C z^{-1/3}$$

$$v'' = \frac{-2}{9} C z^{-4/3}$$

$$\Rightarrow \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b^2 = Q$$

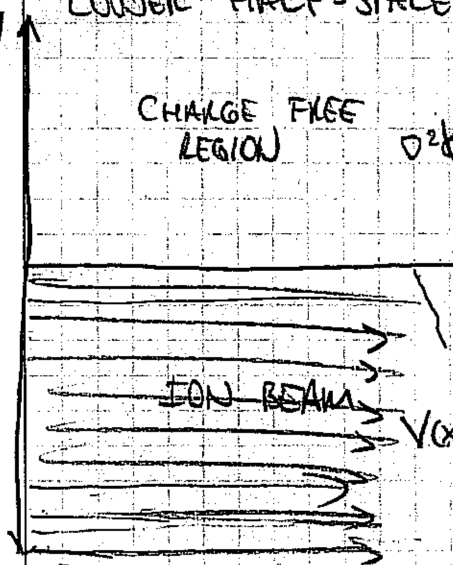
$$\left[\overset{\shortparallel}{\frac{2}{9}\frac{1}{z^2}} \qquad \overset{\shortparallel}{-\frac{1}{9}\frac{1}{z^2}}\right]$$

$$\Rightarrow Q(z) = \frac{1}{9}\frac{r_b^2}{z^2}$$

So Child-Langmuir flow satisfies the PARAXIAL ENVELOPE EQUATION FOR A CONSTANT BEAM RADIUS (AS IT SHOULD!)

PIERCE ELECTRODES: GOING BEYOND PARAXIAL APPROXIMATION

CONSIDER THE CASE A BEAM WHICH FILLS THE LOWER HALF-SPACE.



$\nabla^2 \phi = 0$ $\qquad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

$$V(x) = \left(\frac{J}{\chi}\right)^{2/3} x^{4/3}$$

$$\chi = \left(\frac{4\varepsilon_0}{9}\right)\sqrt{\frac{2q}{m}}$$

FIND SOLUTION SUCH THAT

$$\frac{\partial \phi}{\partial y}(x, y=0) = 0$$

$$\& \quad \phi(x, y=0) = V(x)$$

PIERCE'S SOLUTION: LET THE POTENTIAL BE THE REAL PART OF

$$\phi + iW = V(x+iy) \equiv V(z) \qquad z \equiv x+iy$$

NOTE THAT FOR ANY $V(z)$ WITH derivatives that exist independent of direction (analytic) the real part of $V(z)$ satisfies Laplace's equation: $\dfrac{\partial^2 \mathrm{Re}[V]}{\partial x^2} + \dfrac{\partial^2 \mathrm{Re}[V]}{\partial y^2} = 0$

$$\frac{\partial \phi}{\partial x} = \mathrm{Re}\left[\frac{dV}{dz}\right] \qquad \frac{\partial \phi}{\partial y} = \mathrm{Re}\left[i\frac{dV}{dz}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \qquad \frac{\partial^2 \phi}{\partial y^2} = -\mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] - \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] = 0$$

CHOOSE $V(z) = \left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}$

By construction $\phi(x, y=0) = V(x)$ ✓

$$\phi = \mathrm{Re}\left[\left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}\right]$$

Let $x+iy = r\exp[i\theta]$
$(x+iy)^{4/3} = r^{4/3}\exp\left[\frac{4i\theta}{3}\right]$

$$= \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3}\,\mathrm{Re}\left[\exp\left[i\,\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]\right]$$

$$\boxed{\phi(x,y) = \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3} \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]}$$

Note that $\phi(x,y) = \phi(x,-y) \Rightarrow \frac{\partial\phi}{\partial y}(x, y=0) = 0$

$\phi = 0$ EQUIPOTENTIAL:

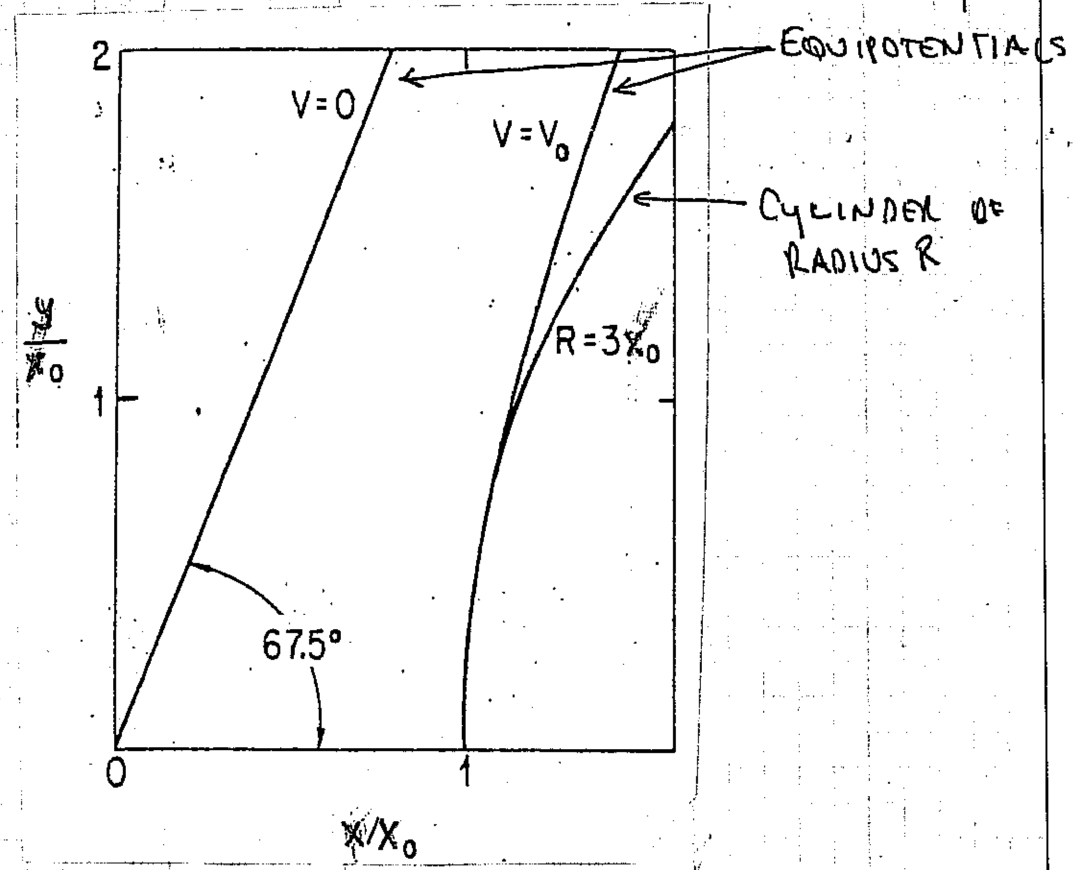
$$\Rightarrow 0 = \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

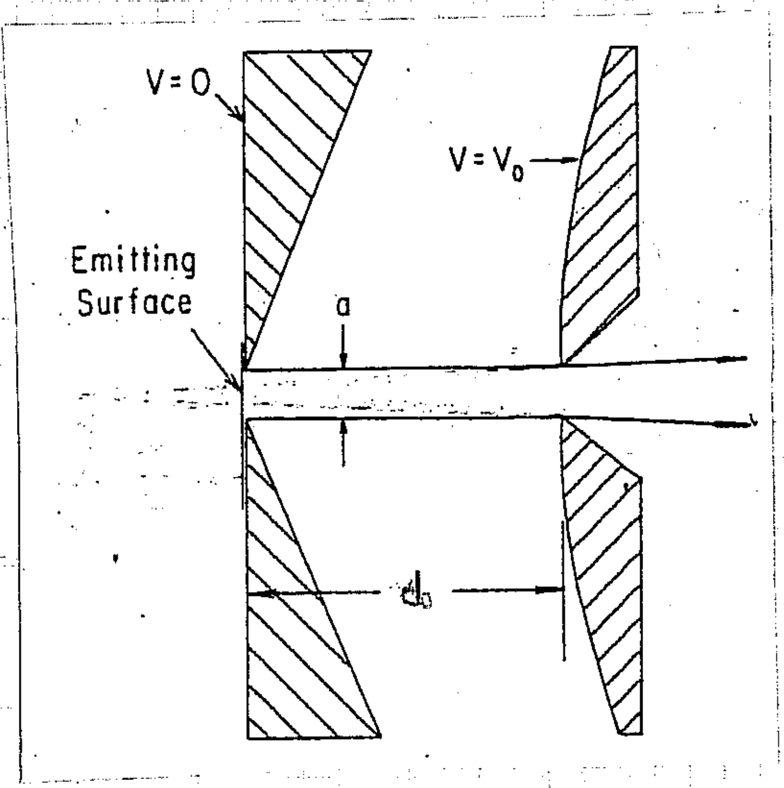
$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{3}{4}\left(\frac{\pi}{2}\right) = 67.5^\circ$$

FOR A GENERAL EQUIPOTENTIAL PASSING THROUGH X_0:

$$x_0^{4/3} = (x^2+y^2)^{2/3} \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

EQUIPOTENTIALS FOR MATCHING TO A PLANAR BOUNDARY

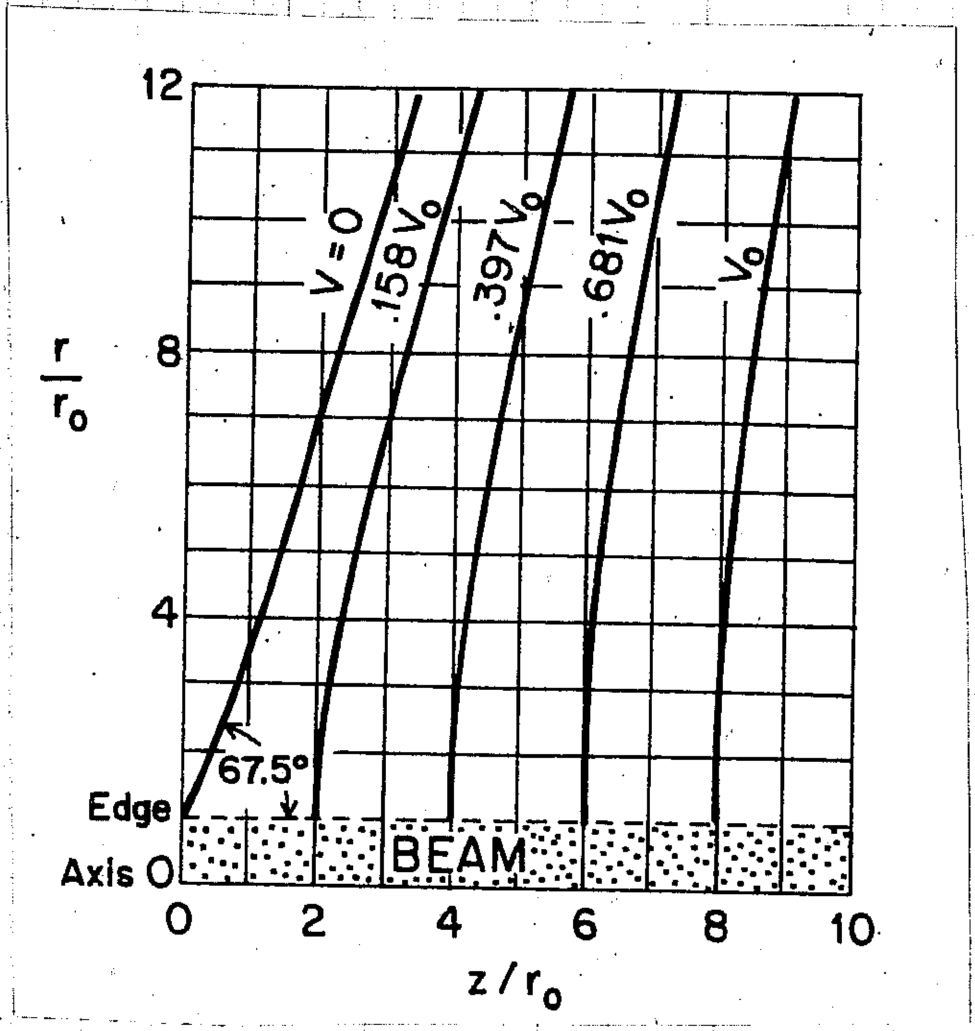




FIGURES FROM "LARGE ION BEAMS, FUNDAMENTALS OF GENERATION AND PROPAGATION" BY A.T. FORRESTER, WILEY, NY, 1988.

ELECTRODES FOR A LONG SLIT ION BEAM SYSTEM

PIERCE ELECTRODES FOR CIRCULAR BEAMS



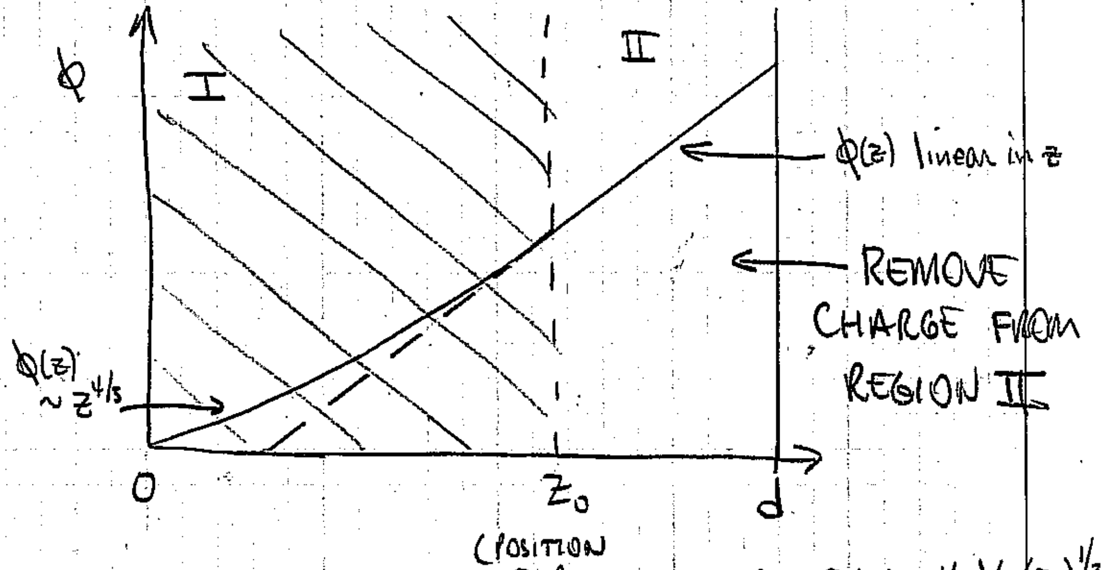
- SOLUTION is similar, but must be done numerically
- $\phi = 0$ is same as planar case

FIGURE FROM A.T. FORRESTER, "LARGE ION BEAMS, ..." Wiley, 1988

TRANSIENTS IN INJECTORS (LAMPEL & TIEFENBACK, Appl. Phys. Lett, 43, 57, 1983)

DURING TURN-ON THERE IS NO SPACE CHARGE IN FRONT OF BEAM, SO FIELDS MAY NOT BE GIVEN BY CHILD-LANGMUIR LAW. ⇒ CURRENT SPIKES POSSIBLE ⇒ ADVERSE TRANSVERSE COUPLING.

SOLUTION: ADJUST VOLTAGE ON DIODE SUCH THAT C-L FIELD OCCURS EVERYWHERE THERE IS BEAM.



$E(z_0) = \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}$

$$\Phi(d) = \int_0^{z_0} E(z)\,dz + \int_{z_0}^{d} E(z)\,dz$$

$$= V_0\left(\frac{z_0}{d}\right)^{4/3} + \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}(d - z_0)$$

$$= V_0\left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

(NOTE V_0 IS THE DESIRED STEADY STATE VOLTAGE ACROSS DIODE).

So if we know $z_0(t)$ we can determine $\Phi(t)$.

$$\frac{1}{2} m \dot{z}_0^{\,2} = q V_0 \left(\frac{z_0}{d}\right)^{4/3}$$

(since by construction, head of beam travels at Child-Langmuir velocity like all particles).

$$\dot{z}_0 = \left(\frac{2qV_0}{m}\right)^{1/2} \left(\frac{z_0}{d}\right)^{2/3}$$

$$\frac{dz_0}{z_0^{2/3}} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{dt}{d^{2/3}}$$

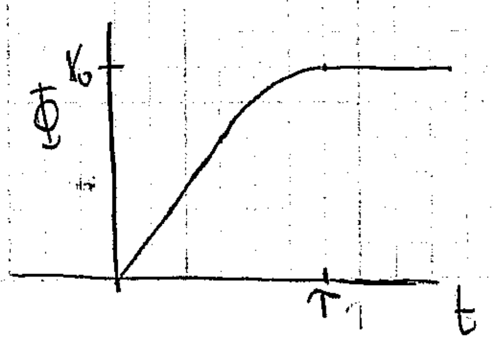
$$\Rightarrow 3 z_0^{1/3} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{t}{d^{2/3}} \quad \Rightarrow \quad t = \frac{3\left(z_0 d^2\right)^{1/3}}{\left(\frac{2qV_0}{m}\right)^{1/2}}$$

Let $\tau = \dfrac{3d}{\left(\frac{2qV_0}{m}\right)^{1/2}}$ = transit time across diode

$$\Rightarrow \frac{t}{\tau} = \left(\frac{z_0}{d}\right)^{1/3}$$

$$\Phi(d, z_0) = V_0 \left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

$$\Rightarrow \Phi(d,t) = \begin{cases} V_0 \left[\frac{4}{3}\left(\frac{t}{\tau}\right) - \frac{1}{3}\left(\frac{t}{\tau}\right)^4\right] & \text{for } 0 < t < \tau \\ V_0 & \text{for } t > \tau \end{cases}$$



INJECTOR CHOICES
(cf. Kwan et al, NIMPRA, 464, 379 (2001))

CHILD-LANGMUIR $\Rightarrow J = \chi \dfrac{V^{3/2}}{d^2}$

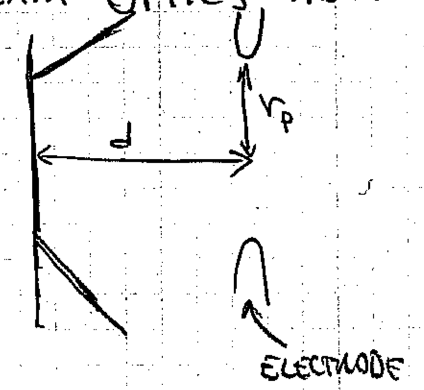
where $\chi = \frac{4}{9}\varepsilon_0\left(\frac{2q}{m}\right)^{1/2}$

SOME CONSTRAINTS:

(1) VOLTAGE BREAKDOWN

EMPIRICALLY $V \lesssim 100\,\text{kV} \begin{cases} \left(\dfrac{d}{1\,\text{cm}}\right) & \text{for } d \leq 1\,\text{cm} \\ \left(\dfrac{d}{1\,\text{cm}}\right)^{1/2} & \text{for } d \geq 1\,\text{cm} \end{cases}$

(2) BEAM OPTICS ABERRATIONS: $d \gtrsim 3$ to $4\ r_p$ (TYPICALLY)



NOTE THAT

$J \sim \dfrac{V^{3/2}}{d^2} \sim \begin{cases} V^{-1/2} & d<1\,\text{cm} \\ V^{-3/2} & d>1\,\text{cm} \end{cases} \qquad I \sim \pi r_p^2 J \sim \left\{ V^{3/2} \right.$

Thus current density decreases with size and voltage, but I increases.

Air $V = 24\,S + 7\sqrt{S}$

from A. Faltens:

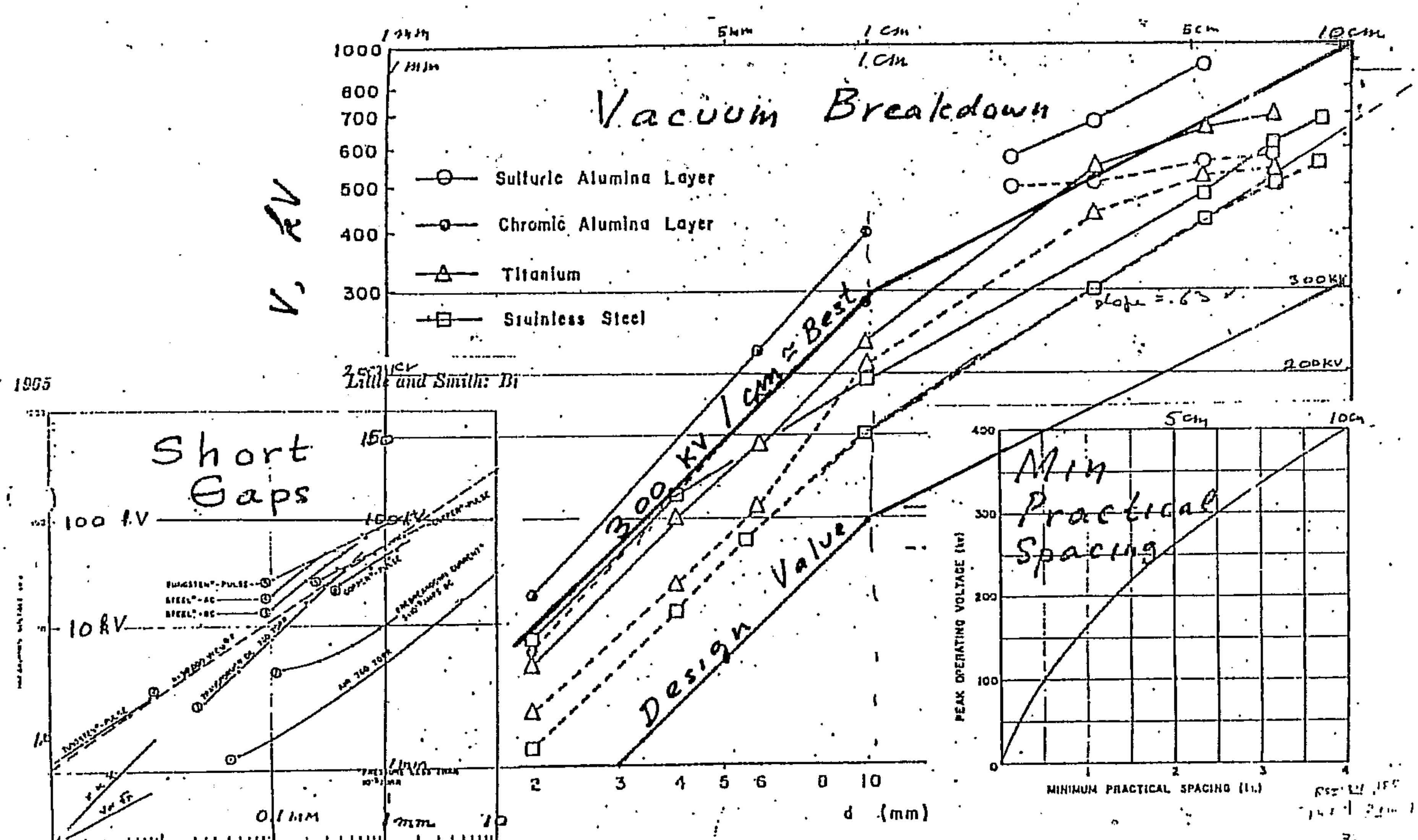


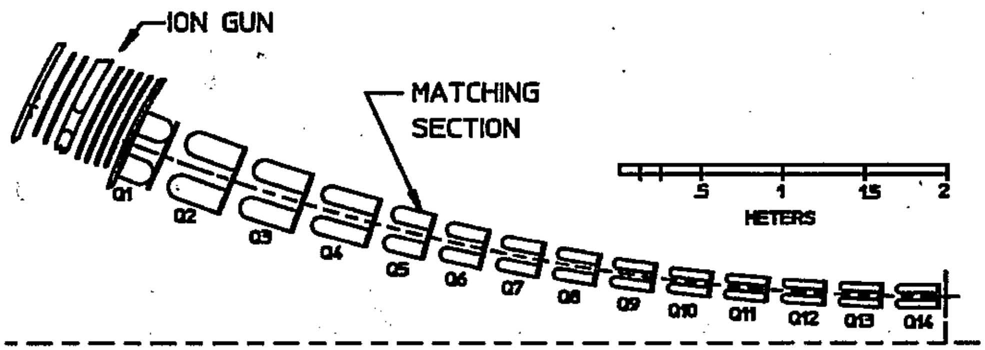
Figure 7.13 Log U - log d plot.

Fig. 1. Breakdown voltage-vs.-gap length for uniform-field and non-uniform-field geometry. Numbers on curves indicate the

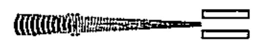
MULTIPLE BEAMLET INJECTORS CAN HAVE HIGHER CURRENT DENSITY

DECREASING SIZE OF INJECTOR

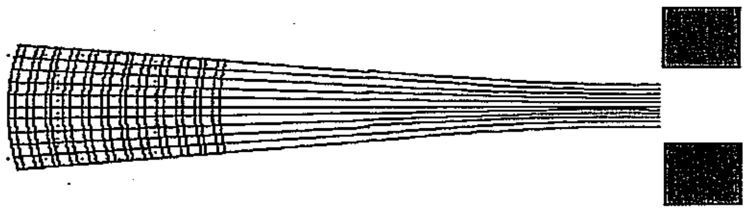
traditional design using single large diameter source



advanced design using multiple beamlets



Each beamlet carries higher current density; But merging beamlets increases thermal spread.



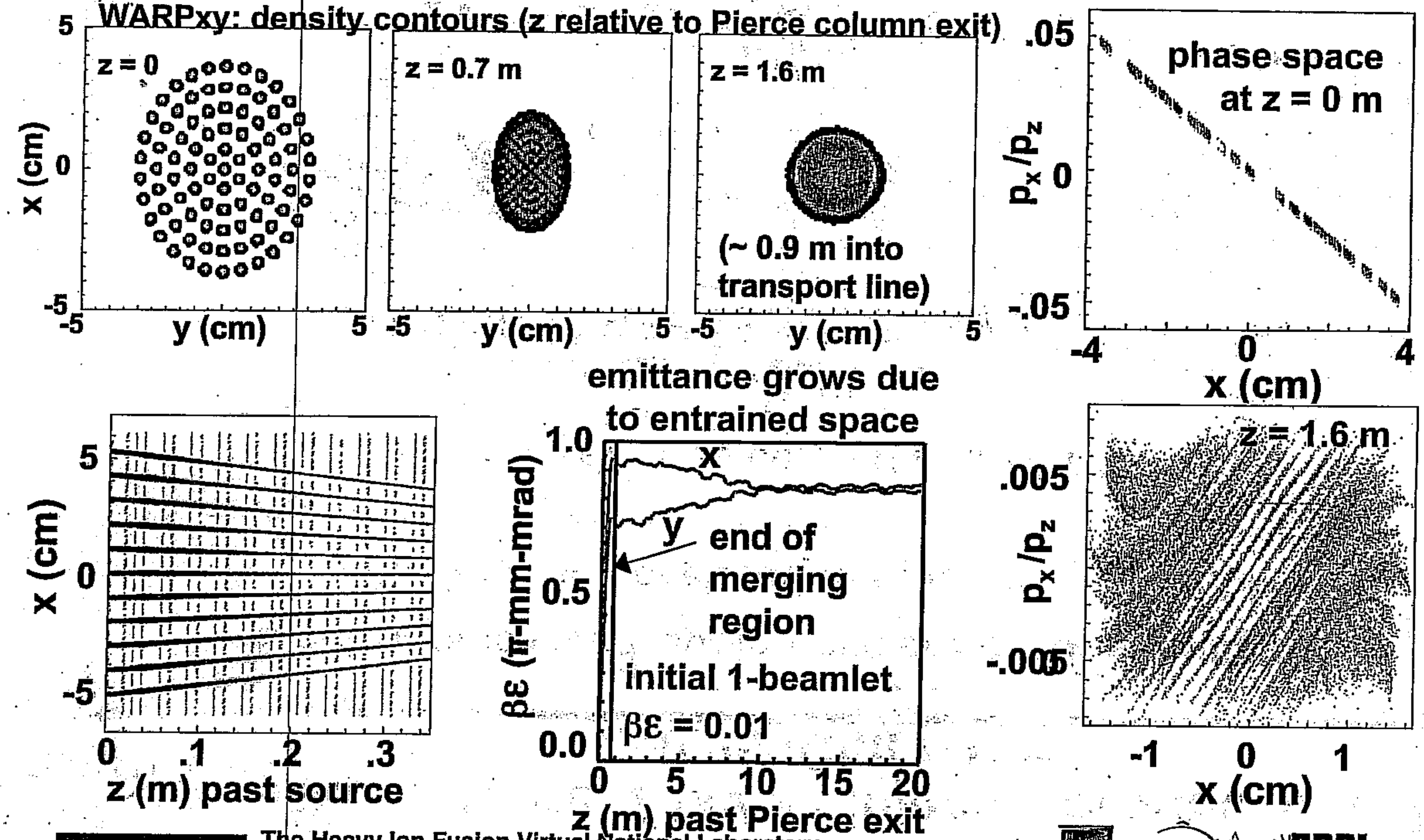
Child-Langmuir $J_{CL} \propto \frac{V^{3/2}}{d^2}$

Breakdown limit $V \propto d^{1.0\ to\ 0.5}$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} J \propto V^{-1/2\ to\ -5/2} \propto d^{-1/2\ to\ -5/4}$$

Merge and match beamlets into an ESQ channel

Simulations of merging-beamlet injector



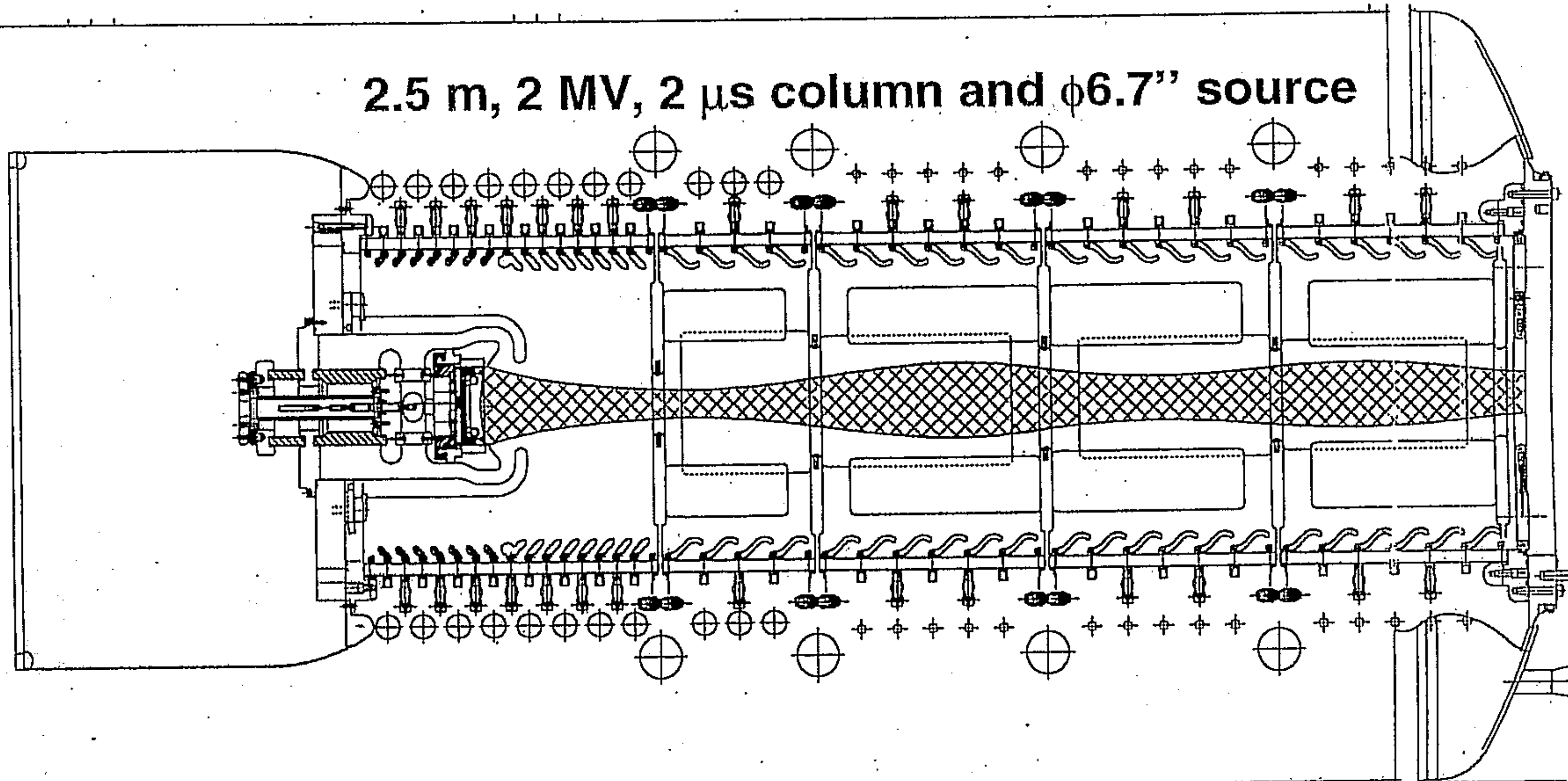
The Heavy Ion Fusion Virtual National Laboratory.

from D.I. Grote, E. Henestroza, J.W. Kwan, "Design & Simulation of the Multibeamlet Injector for a High Current Accelerator" submitted to [illegible] (2002)

0.8 Ampere, 2 MV K^+ Injector produced a λ=0.25μC/m beam

Electrostatic Quadrupole Accelerator for simultaneous focusing and acceleration of ion beams to 2 MV.

2.5 m, 2 MV, 2 μs column and ϕ6.7" source



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SCALING OF BRIGHTNESS IN INJECTORS

$$\epsilon_N = 4\,\beta\,\langle x^2\rangle^{1/2}\langle x'^2\rangle^{1/2} = \frac{4}{c}\left(\frac{r_b}{2}\right)\langle v_x^2\rangle^{1/2}$$

$$\frac{1}{2}mv_x^2 = \frac{1}{2}kT$$

$$\epsilon_N = 2\,r_b\sqrt{\frac{kT}{mc^2}}$$

$$\Rightarrow\quad B = \frac{I}{\epsilon_N^2} = \frac{\pi J}{4(kT/mc^2)} \sim \frac{J}{T}$$

$\Rightarrow$ FOR HIGH BRIGHTNESS & HIGH CURRENT MAY WISH TO ACCELERATE MANY BEAMLETS AND THEN MERGE TO FORM SINGLE BEAM.

MANY ISSUES NOT DISCUSSED HERE!

- SOURCES
- ELECTRON TRAPPING
- CONVERGING BEAMS
- MATCHING TO AN ESQ (e.g.)
- rf
- ...