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NE 290 H

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Injectors and longitudinal physics -- II

1. Acceleration - introduction
2. Space charge of short bunches (rf)
3. Space charge of long bunches
4. Longitudinal space charge waves
5. Longitudinal rarefaction waves and bunch ends

Summary of fluid equations

Let $n(\underline{x}, t) = \int d^3p f(\underline{x}, \underline{p}, t)$ PARTICLE DENSITY

$\underline{v}(\underline{x}, t) \equiv \frac{1}{n(\underline{x}, t)} \int d^3p \frac{\underline{p}}{\gamma m} f(\underline{x}, \underline{p}, t)$ FLUID VELOCITY

$\underline{P}(\underline{x}, t) \equiv \frac{1}{n(\underline{x}, t)} \int d^3p \underline{p} f(\underline{x}, \underline{p}, t)$ FLUID MOMENTUM

$\underline{P}(\underline{x}, t) \equiv \int d^3p (\underline{p} - \underline{P})(\frac{\underline{p}}{\gamma m} - \underline{v}) f(\underline{x}, \underline{p}, t)$ PRESSURE TENSOR

$\frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m}$ $\frac{d\underline{p}}{dt} = q(\underline{E}(\underline{x}, t) + \frac{\underline{p}}{\gamma m} \times \underline{B}(\underline{x}, t))$ $\gamma^2 = \frac{\underline{p} \cdot \underline{p}}{(mc)^2} + 1$

CONTINUITY EQUATION: $\frac{\partial n(\underline{x}, t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x}, t)$

MOMENTUM EQUATION: $\frac{\partial \underline{P}}{\partial t} + \underline{v} \cdot \frac{\partial \underline{P}}{\partial \underline{x}} = q(\underline{E} + \underline{v} \times \underline{B}) - \frac{1}{n(\underline{x}, t)} \frac{\partial}{\partial \underline{x}} \cdot \underline{P}$

THE ABOVE EQUATIONS ARE RELATIVISTICALLY CORRECT,
IN THE NON-RELATIVISTIC LIMIT THE CONTINUITY EQUATION
REMAINS UNCHANGED & THE MOMENTUM EQUATION MAY BE WRITTEN:

NON RELATIVISTIC $\rightarrow \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \frac{\partial \underline{v}}{\partial \underline{x}} = \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B}) - \frac{1}{m n} \frac{\partial}{\partial \underline{x}} \cdot \underline{P}$

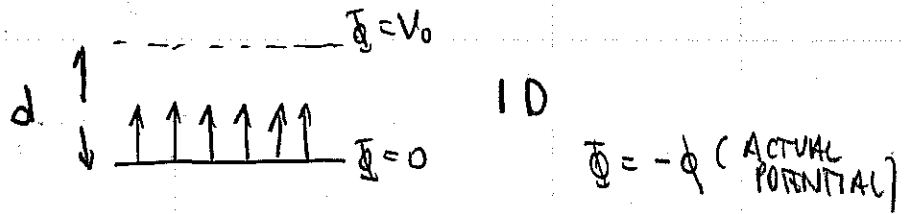
THESE EQUATIONS ARE SUPPLEMENTED WITH MAXWELL'S EQUATIONS:
for $\underline{E}(\underline{x}, t)$ & $\underline{B}(\underline{x}, t)$

$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = \frac{q n(\underline{x}, t)}{\epsilon_0}$ $\frac{\partial}{\partial \underline{x}} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

$\frac{\partial}{\partial \underline{x}} \cdot \underline{B} = 0$ $\frac{\partial}{\partial \underline{x}} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t}$ $\underline{J}(\underline{x}, t) = q n(\underline{x}, t) \underline{v}$

NEED ADDITIONAL EQUATIONS SUCH AS $\underline{P} = 0$ OR ENERGY EQUATION
TO TERMINATE SET OF EQUATIONS.

Summary of Child Langmuir Law



CURRENT DENSITY: $J = \frac{4}{9} \epsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$

$\Phi(z) = V_0 \left(\frac{z}{d}\right)^{4/3}$ - ELECTROSTATIC POTENTIAL

$E(z) = \frac{4}{3} \frac{V_0}{d} \left(\frac{z}{d}\right)^{1/3}$ ELECTRIC FIELD

$v(z) = \left(\frac{2qV_0}{m}\right)^{1/2} \left(\frac{z}{d}\right)^{2/3}$ LONGITUDINAL VELOCITY

$\rho(z) = \frac{J}{v(z)} = \left(\frac{J^2 m}{2qV_0}\right)^{1/2} \left(\frac{z}{d}\right)^{-2/3}$

IF WE MULTIPLY BY πV_b^2 (TO ACCOUNT FOR FINITE RADIUS) / BEAM

$I = \frac{4}{9} \epsilon_0 \left(\frac{2q}{m}\right)^{1/2} \left(\frac{V_b}{d}\right)^2 V_0^{3/2}$

$K \equiv$ Gun permeance $\equiv \frac{I}{V_0^{3/2}}$ [DIMENSIONAL CONSTANT / VOLTAGE^{3/2}]

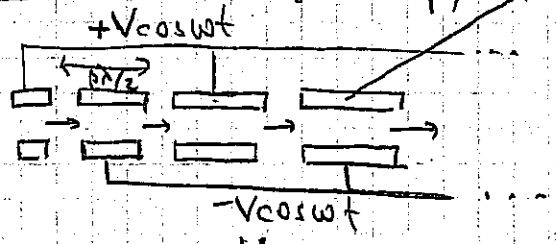
GENERALIZED
(EXAMPLE: Q(z) = ...)
(DIMENSIONLESS)

$Q(z) = \frac{\lambda}{4\pi\epsilon_0 \Phi(z)} = \frac{\pi V_b^2 \rho(z)}{4\pi\epsilon_0 \Phi(z)} = \frac{1}{9} \left(\frac{V_b^2}{z^2}\right)$

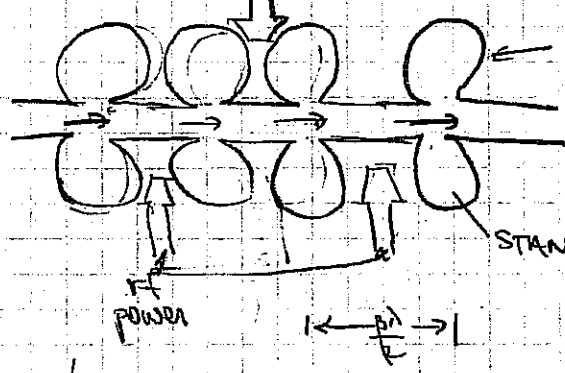
(NOTE THAT CHILD-LANGMUIR LAW ONLY VALID FOR $z \gg \lambda_D$, WHERE $\frac{1}{2} m v(z)^2 \gg kT$ & $\lambda_D = \frac{v_{th}}{\omega_p} = \frac{\sqrt{kT/m}}{\left(\frac{qV(z)}{\epsilon_0 m}\right)^{1/2}}$)

ACCELERATION

rf (radio-frequency)

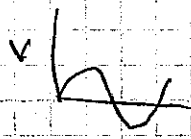


(WIDER GAP LINE)
LOW FREQUENCIES (< 100 MHz)



(COUPLED CAVITY LINE)
 $0.4 < \beta < 1.0$

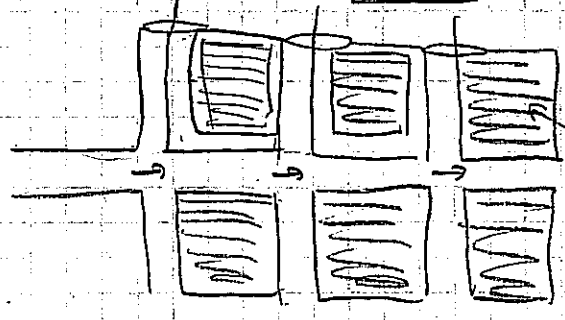
FREQUENCIES ~ 100'S MHz - ~ GHz



IN EACH GAP $E = E_m \sin \omega t$

Induction acceleration

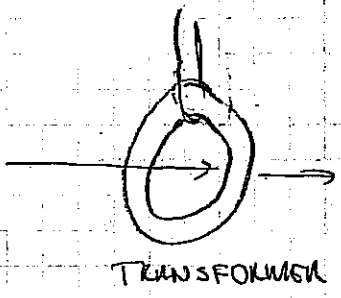
PULSED POWER



(INDUCTION LINEAC)

$\nabla \times E = -\partial B / \partial t$

IN EACH GAP $E = \text{CONSTANT}$
(OR SOME PRESCRIBED FUNCTION)



I STRATEGY FOR CALCULATING LONGITUDINAL EQUATIONS OF MOTION

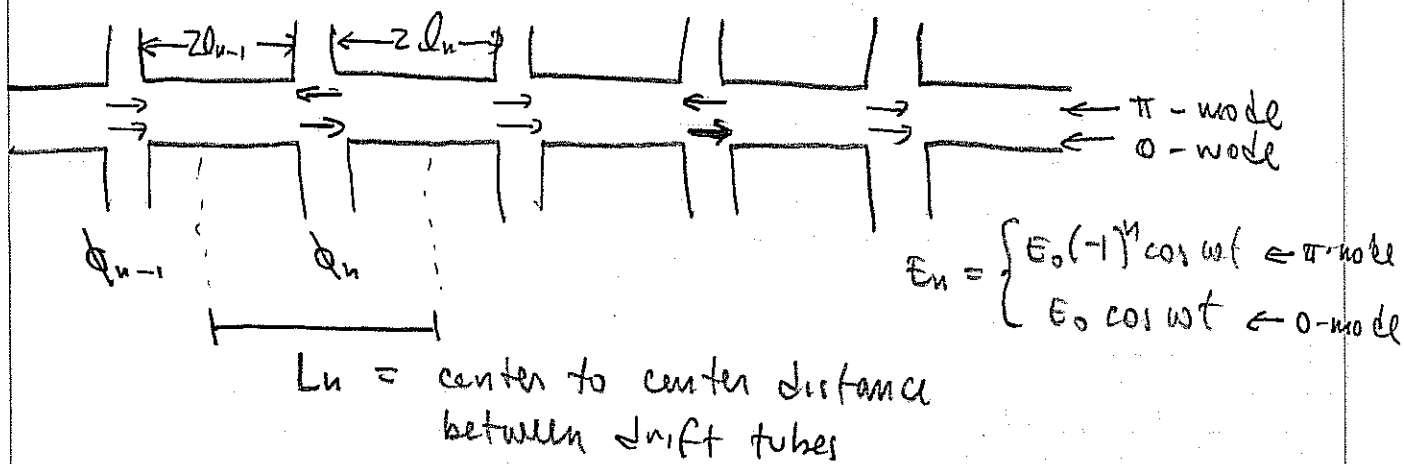
1. rf / short bunches

- EXTERNAL FIELD: ACCELERATES AND FOCUSES
 - CALCULATE CHANGE IN TIME AND ENERGY AS PARTICLE MOVES FROM ACCELERATING GR TO ACCELERATING GAP
 - APPROXIMATE MOTION AS CONTINUOUS
- SPACE CHARGE FIELD: UNIFORM DENSITY ELLIPSOID

2. INDUCTION / LONG BUNCHES

- EXTERNAL FIELD: ACCELERATION, FOCUSING, AND COMPRESSION ACCOMPLISHED BY PRESCRIBING VOLTAGE WAVE FORM
- FOCUSING (CONFINEMENT) IS DONE AT BEAM ENDS
- SPACE CHARGE FIELD: $E_z \propto \frac{\partial \lambda}{\partial z}$

RF longitudinal equation of motion



$E_z = E_0 \cos(\phi_s)$ ← synchronous particle enters each gap at same phase

RESONANCE CONDITION ON SYNCHRONOUS PARTICLE:

$$L_{n-1} = \frac{\beta_s \lambda}{2} \begin{cases} \frac{1}{2} \\ 1 \end{cases} \begin{matrix} \pi\text{-mode} \\ 0\text{-mode} \end{matrix}$$

$\lambda = \frac{2\pi c}{\omega} =$ light travel distance in one cycle of oscillation

(IT TAKES $\frac{1}{\beta_s}$ OSCILLATION PERIOD TO TRAVEL BETWEEN GAPS),

$\beta_s = \frac{v_s}{c} =$ velocity of synchronous particle

PARTICLE PHASE RELATIVE TO rf at the n^{th} gap:

$$\phi_n = \phi_{n-1} + \omega \frac{2L_{n-1}}{\beta_{n-1} c} + \begin{cases} \pi & \pi\text{-mode} \\ 0 & 0\text{-mode} \end{cases}$$

$$\Delta(\phi - \phi_s)_n = 2\pi \beta_{s,n-1} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right) \begin{cases} \frac{1}{2} \\ 1 \end{cases} \begin{matrix} \pi\text{-mode} \\ 0\text{-mode} \end{matrix}$$

$$\approx -2\pi \frac{\delta\beta}{\beta_{s,n-1}} \begin{cases} \frac{1}{2} \\ 1 \end{cases}$$

A VELOCITY DIFFERENCE LEADS TO A PHASE DIFFERENCE!!

$$\Delta(\phi - \phi_s)_n \approx -2\pi \frac{W_{n-1} - W_{s,n-1}}{m c^2 \gamma_{s,n-1}^3 \beta_{s,n-1}} \begin{cases} \frac{1}{2} \\ 1 \end{cases}$$

$$W = (\gamma - 1) m c^2$$

$$\frac{1}{\beta} - \frac{1}{\beta_s} \approx -\frac{\delta\beta}{\beta_s^2}$$

$$\delta W = \gamma_s^3 \beta_s m c^2 \delta\beta$$

SIMILARLY, A PHASE DIFFERENCE PRODUCES

AN ENERGY CHANGE (RELATIVE TO SYNCHRONOUS PARTICLES)

$$\Delta(W - W_s)_n = q E_0 L_n (\cos \psi_n - \cos \psi_{s,n})$$

$$L_n = \frac{(\beta_{s,n-1} + \beta_{s,n}) \lambda}{2} \left\{ \begin{matrix} 1/2 \\ 1 \end{matrix} \right\} = \text{CENTER-TO-CENTER DISTANCE BETWEEN DRIFT SECTIONS}$$

$$(\Delta W_s = q E_0 L_n \cos \psi_s)$$

CAMPAD

ENERGY (VELOCITY) DIFFERENCE \Rightarrow
ARRIVAL TIME DIFFERENCE
(PHASE DIFFERENCE)



PHASE DIFFERENCE IN π
FIELD \Rightarrow DIFFERENCE
IN ENERGY GAIN



CONVERTING TO A CONTINUOUS VARIABLE:

$$\Delta(\phi - \phi_s) \rightarrow \frac{d\Delta\phi}{dn} \quad \Delta(W - W_s) \rightarrow \frac{d\Delta W}{dn}$$

$$\Rightarrow \left[\gamma_s^3 \beta_s^3 \frac{d\Delta\phi}{ds} = -2\pi \frac{\Delta W}{mc^2 \lambda} \right] \quad \frac{dn}{ds} = \frac{1}{\beta_s \lambda} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$$

$$\frac{d\Delta W}{ds} = qE_0 (\cos \phi - \cos \phi_s)$$

$$\frac{d}{ds} \left[\gamma_s^3 \beta_s \frac{d\Delta\phi}{ds} \right] = -2\pi \frac{qE_0}{mc^2 \lambda} [\cos \phi - \cos \phi_s] \quad (I)$$

NOW THE SPATIAL SEPARATION IS GIVEN BY:

$$\Delta z \equiv z - z_s = -\frac{\beta_s \lambda}{2\pi} \Delta\phi$$

$$\Rightarrow \frac{d}{ds} \text{ ALSO, LET } \cos \phi - \cos \phi_s \approx -\sin \phi_s \Delta\phi \quad \left[\text{for } \frac{2\pi \Delta z}{\beta_s \lambda} = \Delta\phi \ll 1 \right]$$

$$\Rightarrow \frac{d}{ds} \left[\gamma_s^3 \beta_s \frac{d}{ds} \left(\frac{\Delta z}{\beta_s} \right) \right] \approx -\frac{2\pi}{\lambda} \frac{qE_0}{mc^2} \sin \phi_s \frac{\Delta z}{\beta_s}$$

WHEN THE ACCELERATION RATE IS SMALL

$$\Rightarrow \frac{d^2}{ds^2} \Delta z \approx -\frac{2\pi}{\lambda} \frac{qE_0 \sin \phi_s}{\gamma_s^3 \beta_s mc^2} \Delta z$$

$$\equiv -k_{s0}^2 \Delta z \quad (\text{synchrotron oscillations})$$

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RETURNING TO $\Delta W - \phi$ NOTATION

Let $w = \frac{\Delta W}{mc^2}$

$A = \frac{2\pi}{\beta_s^3 \gamma_s^3 \lambda}$

$B = \frac{q E_0}{mc^2}$

$$\Rightarrow w' = B(\cos \phi - \cos \phi_s)$$

$$\phi' = -Aw$$

$$\phi'' = -AB(\cos \phi - \cos \phi_s)$$

MULTIPLYING BY ϕ' AND INTEGRATING:

$\frac{\phi'^2}{2} = -AB(\sin \phi - \phi \cos \phi_s) + \text{const}$

Using $\phi' = -Aw$ & DIVIDING BY A

$\Rightarrow \frac{Aw^2}{2} + B(\sin \phi - \phi \cos \phi_s) = \text{CONST.}$

kinetic energy

potential energy

$\frac{dW_s}{ds} \sim qE_0 \cos \phi_s$

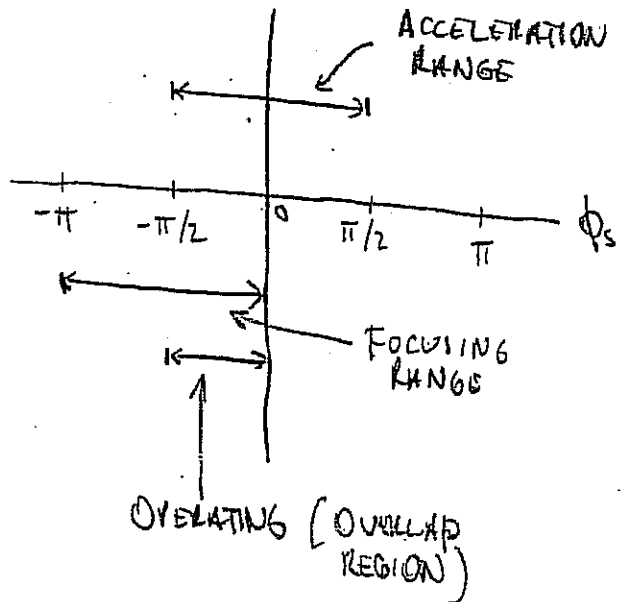
$V(\phi) = B(\sin \phi - \phi \cos \phi_s)$

$\frac{dV}{d\phi} = B(\cos \phi - \cos \phi_s)$

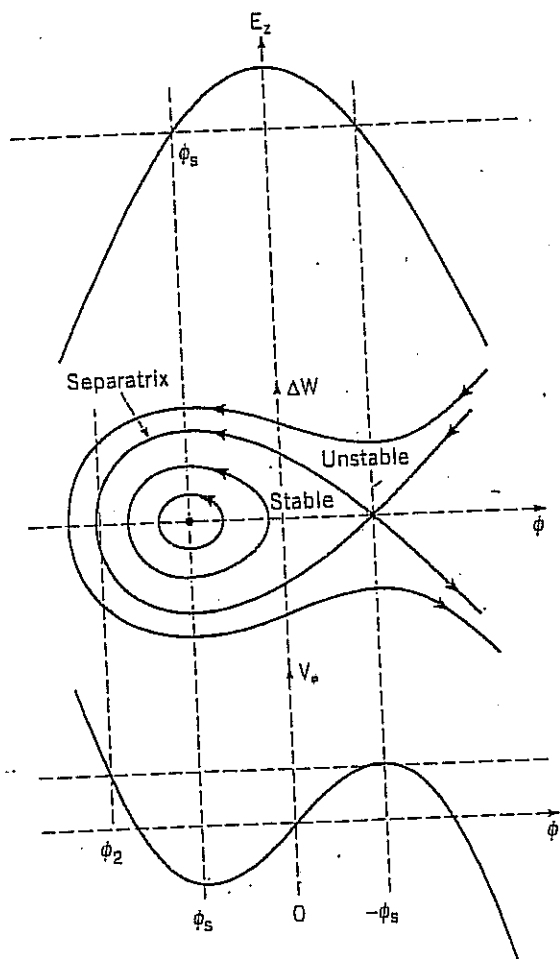
$\frac{d^2V}{d\phi^2} = -B \sin \phi$

$> 0 \Rightarrow -\pi < \phi_s < 0$

FOR LONGITUDINAL FOCUSING



simultaneous acceleration and a potential well when $-\pi/2 \leq \phi_s \leq 0$. The stable region for the phase motion extends from $\phi_2 < \phi < -\phi_s$, where the lower phase limit ϕ_2 can be obtained numerically by solving for ϕ_2 using $H_\phi(\phi_2) = H_\phi(-\phi_s)$. Figure 6.3 shows longitudinal phase space and the longitudinal potential well. At the potential maximum, where $\phi = -\phi_s$, we



from T. Waugler's "PRINCIPLES OF RF LINEAR ACCELERATORS"

w-phi PHASE SPACE

Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$ and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

Electrostatic potential of a uniform density

ellipsoid in free space

(cf Landau & Lifshitz, Classical Theory of Fields, p. 217)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

It can be shown that:

$$\phi = \frac{-\rho}{4\epsilon_0} a b c \int_{s_{min}}^{\infty} \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s} \right) \frac{ds}{R_s}$$

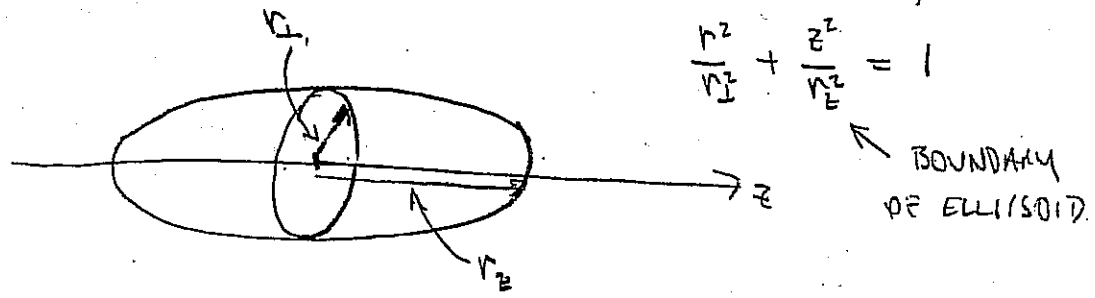
$$R_s = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

$$s_{min} = \begin{cases} 0 & \text{if interior point } \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right) \\ \xi & \text{if exterior point } \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} > 1 \right) \end{cases}$$

Here ξ is the positive root of

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1$$

AXISYMMETRIC SPACE-CHARGE FIELD OF A BUNCHED BEAM



INTERIOR

THE POTENTIAL OF A UNIFORM DENSITY BUNCH IN FREE SPACE (A MACLAURIN SPHEROID) IS GIVEN BY:

(cf Landau & Lifshitz, Classical Theory of ~~Fluids~~ Fluids, p 297)

$$\phi = \frac{-\rho}{4\epsilon_0} (\alpha_{\perp} r_{\perp}^2 + \alpha_{\parallel} z^2 - \delta)$$

where $\alpha_{\perp} = r_{\perp}^2 r_z \int_0^{\infty} \frac{ds}{(r_{\perp}^2 + s) \Delta}$

$$\alpha_{\parallel} = r_{\perp}^2 r_z \int_0^{\infty} \frac{ds}{(r_z^2 + s) \Delta}$$

$$\delta = r_{\perp}^2 r_z \int_0^{\infty} \frac{ds}{\Delta}$$

where $\Delta^2 = (r_{\perp}^2 + s)^2 (r_z^2 + s)$

FOR NON-RELATIVISTIC BEAM:

$$E_z = -\frac{\partial \phi}{\partial z} = f \frac{\rho}{\epsilon_0} z$$

$$E_r = -\frac{\partial \phi}{\partial r} = \frac{(1-f)}{2} \frac{\rho}{\epsilon_0} r$$

$$f = f(\alpha) = \begin{cases} \frac{\alpha^2}{1-\alpha^2} \left[\frac{1}{\sqrt{1-\alpha^2}} \tanh^{-1} \sqrt{1-\alpha^2} - 1 \right] & \alpha < 1 \\ \frac{1}{3} & \alpha = 1 \\ \frac{\alpha^2}{\alpha^2-1} \left[1 - \frac{1}{\sqrt{\alpha^2-1}} \tanh^{-1} \sqrt{\alpha^2-1} \right] & \alpha > 1 \end{cases} \quad \alpha \equiv \frac{r_{\perp}}{r_z}$$

THE FIELD FOR ALL RADII MAY BE WRITTEN:

$$E_r = \frac{\rho}{2\epsilon_0} \left[\frac{\alpha^2}{(\alpha^2 + \chi)(1 + \chi)^{3/2}} - F(\chi, \alpha) \right] r$$

$$E_z = \frac{\rho}{\epsilon_0} [F(\chi, \alpha)] z$$

$$F(\chi, \alpha) = \begin{cases} \frac{\alpha^2}{1 - \alpha^2} \left[\frac{1}{\sqrt{1 - \alpha^2}} \tanh^{-1} \frac{\sqrt{1 - \alpha^2}}{\sqrt{1 + \chi}} - \frac{1}{\sqrt{1 + \chi}} \right] & \alpha < 1 \\ \frac{1}{3(1 + \chi)^{3/2}} & \alpha = 1 \\ \frac{\alpha^2}{\alpha^2 - 1} \left[\frac{1}{\sqrt{1 + \chi}} - \frac{1}{\sqrt{\alpha^2 - 1}} \tanh^{-1} \left(\frac{\sqrt{\alpha^2 - 1}}{\sqrt{1 + \chi}} \right) \right] & \alpha > 1 \end{cases}$$

χ satisfies:

$$\frac{\alpha^2 (r^2/r_L^2)}{\alpha^2 + \chi} + \frac{(z^2/r_E^2)}{1 + \chi} = 1 \quad \text{for exterior particle}$$

$$\chi = 0 \quad \text{for interior particle}$$

$$\alpha = \frac{r_L}{\gamma r_z}$$

FOR AN EXTERIOR PARTICLE AT r & $z \Rightarrow \chi$ CAN BE SOLVED FOR (QUADRATIC EQUATION FOR χ).

$\Rightarrow E_r$ & E_z ARE KNOWN ANALYTICALLY FOR ALL r, z .

EXAMPLE: $r = 0 \Rightarrow \chi = \frac{z^2}{r_z^2} - 1$

RELATIVISTIC TRANSFORMATION FROM BEAM FRAME TO LAB FRAME:
(see e.g. JACKSON, CLASSICAL ELECTRODYNAMICS)

$$\underline{E} = \gamma (\underline{E}' - c\beta \times \underline{B}') - \frac{\gamma^2}{\gamma+1} \beta (\beta \cdot \underline{E}')$$

$$\underline{B} = \gamma (\underline{B}' + \frac{\beta}{c} \times \underline{E}') - \frac{\gamma^2}{\gamma+1} \beta (\beta \cdot \underline{B}')$$

$$\underline{B}' = 0$$

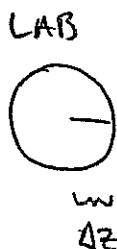
$$\Rightarrow E_z = \gamma E'_z - \frac{\gamma^2}{\gamma+1} \beta^2 E'_z = E'_z$$

$$E_r = \gamma E'_r$$

$$B_\theta = \frac{\gamma\beta}{c} E'_r$$

$$F_z = q E_z = q E'_z$$

$$F_r = q (E_r - v_z B_\theta) = \frac{1}{\gamma} q E'_r$$



$$\Delta z = \frac{1}{\gamma} \Delta z'$$

$$\underline{x}_\perp = \underline{x}'_\perp$$

$$p = \gamma p'$$

$$\underline{F}_\perp = \frac{d\underline{p}_\perp}{dt} = \gamma_s^3 m v_{zs}^2 \frac{d^2 \underline{x}_\perp}{ds^2} \quad (\text{NEGLECTING } \frac{d\gamma_s}{dt}, \frac{dv_{zs}}{dt})$$

$$\Delta F_z = \frac{d p_{||}}{dt} - \frac{d p'_{||}}{dt} = \gamma_s^3 m v_{zs}^2 \frac{d^2 (z - z_s)}{ds^2}$$

$$(\text{USING } \frac{d}{ds} \gamma \beta = \gamma^3 \frac{d\beta}{ds})$$

FOR RELATIVISTIC BEAM

(cf. BARNARD & LUND 1997
LUND & BARNARD 1997
PAC 97 CONF PROCEEDINGS)

$$\frac{d^2 \underline{x}_L}{ds^2} = \frac{F_L}{\gamma_s^3 \beta_s^2 m c^2}$$

$$\underline{F}_{\perp s} = \frac{q\rho}{2\gamma_s^2 \epsilon_0} [1 - f(\alpha)] \underline{x}_L$$

$$\frac{d^2 \Delta z}{ds^2} = \frac{F_z}{\gamma_s^3 \beta_s^2 m c^2}$$

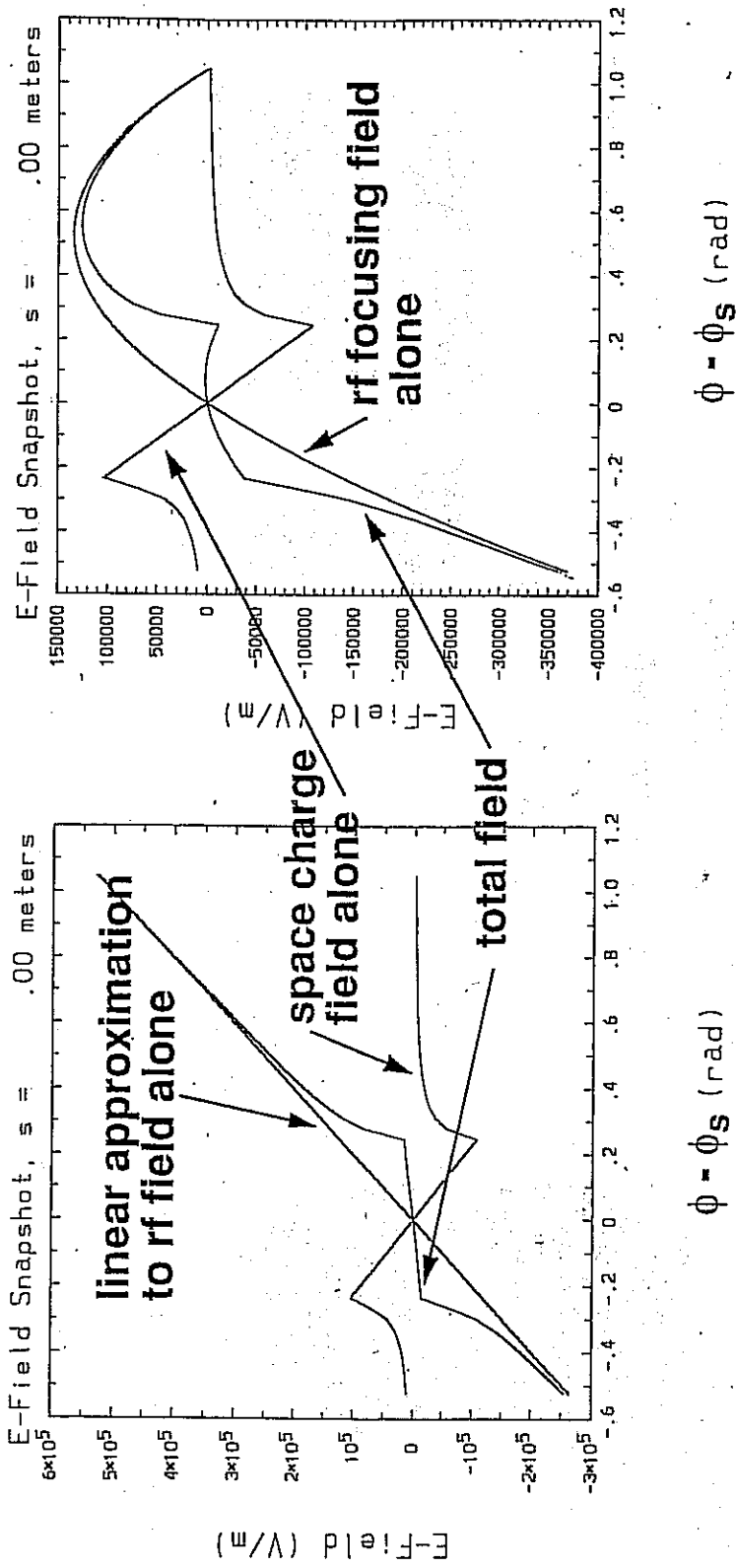
$$F_{zs} = \frac{q\rho}{\epsilon_0} f(\alpha) \Delta z$$

$$\alpha = \frac{r_L}{\gamma_s v_z} \quad \left[\alpha = \frac{r_L}{(v_z \text{ in comoving frame})} \right]$$

COMBINING FOCUSING + SELF FIELDS

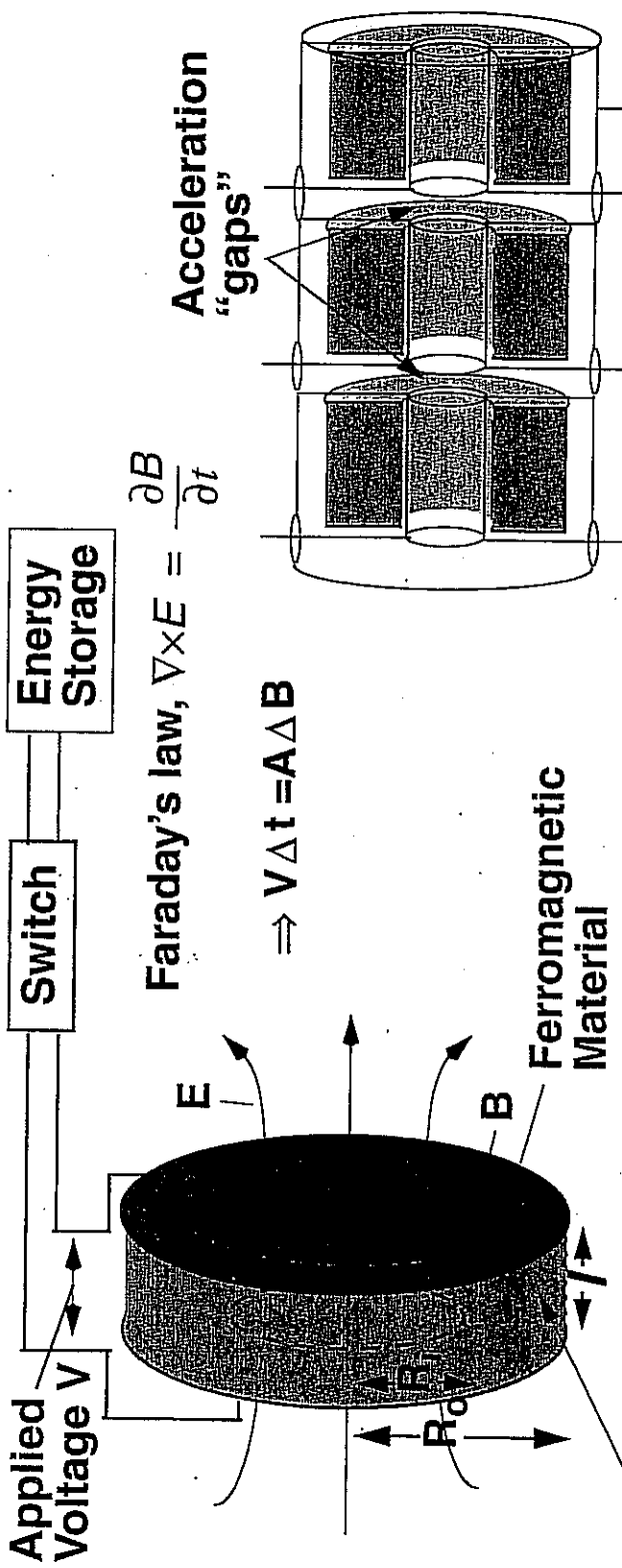
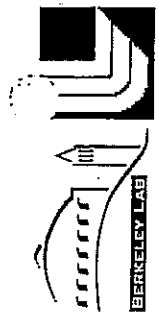
$$\frac{d^2 \Delta z}{ds^2} = -\frac{1}{f_{s0}} \Delta z + \frac{q\rho f(\alpha)}{\gamma_s^3 \beta_s^2 m c^2 \epsilon_0} \Delta z \quad (\text{LINEAR RF})$$

Total field seen by particle is sum of rf and spacecharge



here $\phi - \phi_s = - (2 \pi / \beta_s \lambda) \Delta z$, where $\beta_s c$ is the longitudinal velocity of the synchronous particle and $\lambda = c/v$ is the rf vacuum wavelength

Induction acceleration



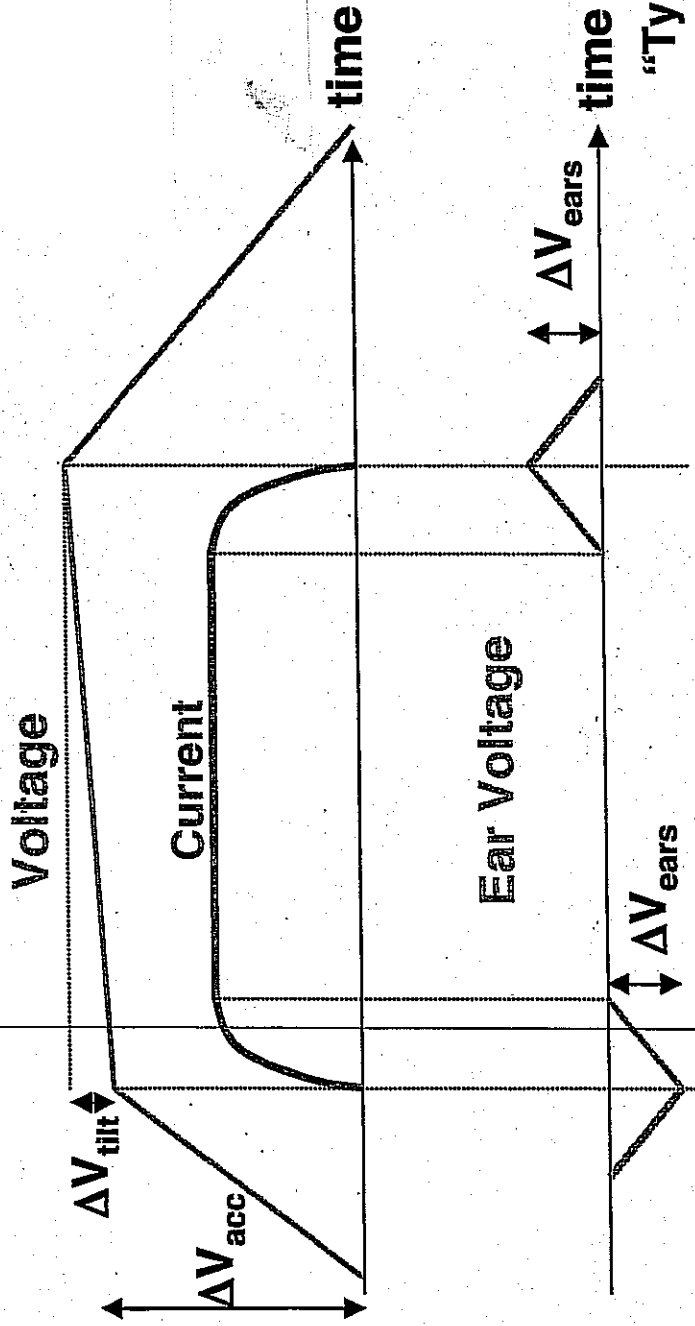
Faraday's law, $\nabla \times E = \frac{\partial B}{\partial t}$
 $\Rightarrow V \Delta t = A \Delta B$

Cross-sectional area A
 $A = (R_o - R_i) l$

Volt-seconds per m: $(dV/dz) \Delta t = (R_o - R_i) \Delta B$ $f_{\text{radial}} f_{\text{longit.}}$
 $\sim 1 \text{ m} \sim 2.5 \text{ T} \sim 0.8 \sim 0.8$

$(dV/dz) \Delta t < \sim 1.6 \text{ V-s/m}$

Several types of waveform are needed to accelerate, compress, and confine the beam



"Typical" numbers:

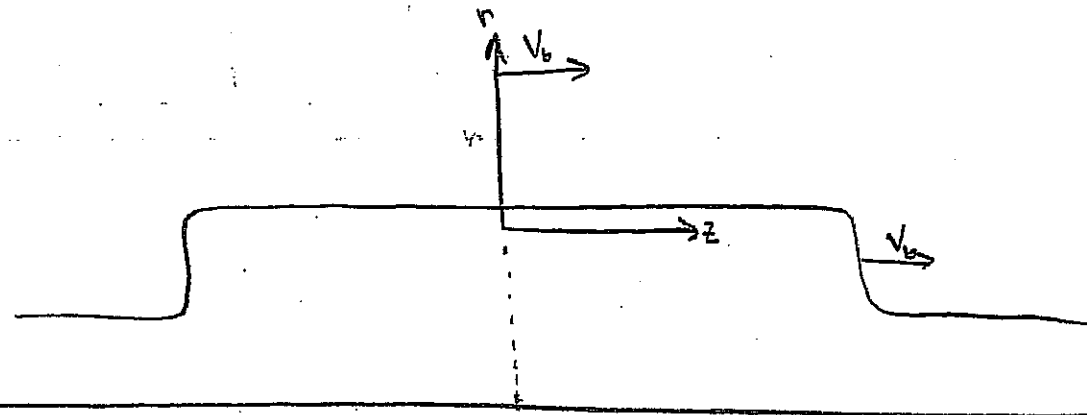
- $\Delta V_{tilt} \sim 1 \text{ kV}$
- $\Delta V_{ears} \sim 14 \text{ kV}$
- $\Delta V_{acc} \sim 100 \text{ kV}$



The Heavy Ion Fusion Virtual National Laboratory



COORDINATE SYSTEM



42-102 100 SHEETS
EST National Brand
Made in U.S.A.

$s=0$

$s = v_b t$ for drifting beam
= position of beam center in lab frame

$s \leftrightarrow t$ are related by βc for drifting beam

z = longitudinal coordinate in beam frame ($z=0$ = beam center)

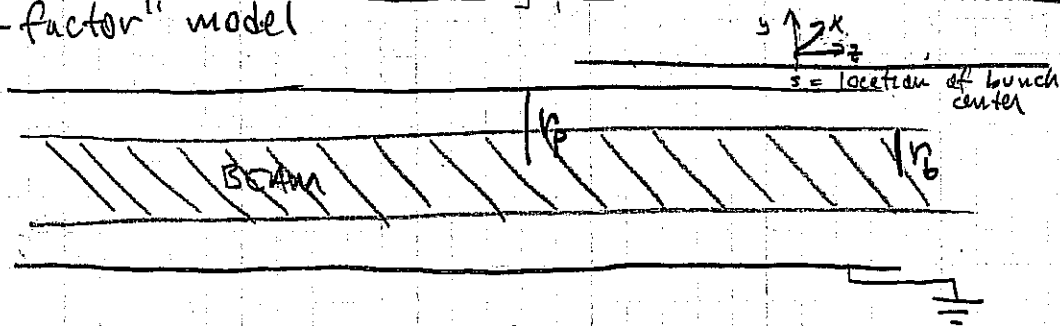
r = radial coordinate in beam frame (or lab frame).

(This class will assume non-relativistic dynamics)

These are ions with $\beta < 0.2$.

LONGITUDINAL PHYSICS OF LONG PULSES (BUNCH LENGTH $\gg r_{pipe}$)

"g-factor" model



If $\frac{\partial^2 \phi}{\partial z^2} \ll \frac{1}{r} \left(\frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} \right) \Rightarrow \frac{\partial \phi}{\partial r} = \frac{-\lambda(r)}{2\pi\epsilon_0 r}$

Let $\rho = \begin{cases} \rho_0 & 0 < r < r_b \\ 0 & r_b < r < r_p \end{cases} \Rightarrow \lambda = \lambda_0 \left(\frac{r}{r_b} \right)^2$

$\phi = \int \frac{\partial \phi}{\partial r} dr = \begin{cases} \frac{\lambda}{2\pi\epsilon_0} \left[\frac{1}{2} \left(1 - \frac{r^2}{r_b^2} \right) + \ln \frac{r_p}{r_b} \right] & 0 < r < r_b \\ \frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{r_p}{r} \right) & r_b < r < r_p \end{cases}$

$\frac{\partial \phi}{\partial z} = \frac{1}{2\pi\epsilon_0} \left[\frac{1}{2} \left(1 - \frac{r^2}{r_b^2} \right) + \ln \frac{r_p}{r_b} \right] \frac{\partial \lambda}{\partial z} - \frac{1}{2\pi\epsilon_0} \left[1 - \frac{r^2}{r_b^2} \right] \frac{\lambda}{r_b} \frac{\partial r_b}{\partial z}$

If $\rho = \text{const} \Rightarrow \frac{\lambda}{r_b^2} = \text{const} \quad \frac{\partial \lambda}{\partial z} = -\frac{2\lambda}{r_b} \frac{\partial r_b}{\partial z}$

$\Rightarrow \frac{\partial \phi}{\partial z} = \frac{1}{2\pi\epsilon_0} \ln \left(\frac{r_p}{r_b} \right) \frac{\partial \lambda}{\partial z}$

$E_z = \frac{-g}{4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$

where $g = 2 \ln \left(\frac{r_p}{r_b} \right)$

[SPACE-CHARGE DOMINATED BEAM]

[Example of $\rho = \text{const}$.
Magnetic Quad focusing
 $\frac{\lambda}{4\pi\epsilon_0 Va} \approx k_{p0}^2 a$
 $\Rightarrow \rho \sim V k_{p0}^2 \approx \text{const}$]

(for space-charge dominated beam)

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FOR EMITTANCE DOMINATED BEAMS:

RADIUS NOT DETERMINED BY λ

$$\text{so } \frac{\delta r_b}{\delta z} \approx 0$$

$$\left\langle \frac{\delta \phi}{\delta z} \right\rangle = \frac{1}{2\pi\epsilon_0} \left[\frac{1}{2} \left(1 - \left\langle \frac{v^2}{v_b^2} \right\rangle \right) + \ln \frac{r_p}{r_b} \right] \frac{\delta \lambda}{\delta z}$$

\parallel
 $1/2$

$$\Rightarrow g = 2 \ln \left(\frac{r_p}{r_b} \right) + \frac{1}{2} \quad (\text{EMITTANCE DOMINATED BEAMS})$$

(SEE REISER, SECTION 6.3 FOR DISCUSSION ON g-FACTOR).

Vlasov - equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial f}{\partial z'} = 0$$

Let $\tilde{f}(z, z', s) \equiv \iiint f \, dx \, dx' \, dy \, dy'$

INTEGRATING VLASOV EQUATION:

If $z'' \neq f(x, x', y, y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \underbrace{\iiint x' \frac{\partial f}{\partial x} \, dx \, dx' \, dy \, dy'}_{=0} + \dots + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0} \quad \text{1D Vlasov}$$

Now let $\lambda \equiv q \int \tilde{f} \, dz'$; $\lambda \bar{z} = \int \tilde{f} z' \, dz'$; $\lambda \bar{z}^2 = \int \tilde{f} z'^2 \, dz'$

Also, let $\Delta z^2 \equiv \bar{z}^2 - (\bar{z}')^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER z' :

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z} (\lambda \bar{z}') = 0} \quad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY z' & INTEGRATING VERSION OF z' :

$$\frac{\partial}{\partial s} \lambda \bar{z}' + \frac{\partial}{\partial z} \lambda \bar{z}'^2 - \lambda \bar{z}'' = 0$$

DIVIDING BY λ , USING CONTINUITY EQUATION & DEFINITION OF Δz^2 :

$$\boxed{\underbrace{\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial}{\partial z} \bar{z}'}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda \Delta z^2)}_{\text{PRESSURE TERM}} = \underbrace{\bar{z}''}_{\text{FORCE}}} \quad \text{(MOMENTUM EQUATION)}$$

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COMBINING g-factor model with fluid equations

$$\frac{\partial \lambda}{\partial s} + \frac{\partial (\lambda \bar{z}')}{\partial z} = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \underbrace{\frac{1}{\lambda} \frac{\partial (\lambda \Delta \bar{z}'^2)}{\partial z}}_{\text{PRESSURE TERM}} = \underbrace{\frac{-g}{4\pi \epsilon_0 m v_0^2} \frac{\partial \lambda}{\partial z}}_{\text{SPACE CHANGE TERM}}$$

WHEN PRESSURE TERM \ll SPACE CHANGE TERM,

(LET $c_s^2 \equiv \frac{g \lambda_0}{4\pi \epsilon_0 m} = \text{"(SPACE CHANGE WAVE SPEED)"}$)

$$\begin{aligned} \Rightarrow \frac{\partial \lambda}{\partial s} + \lambda \frac{\partial \bar{z}'}{\partial z} + \bar{z}' \frac{\partial \lambda}{\partial z} &= 0 \\ \frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{c_s^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} &= 0 \end{aligned}$$

(g1)

LINEARIZING g1

Let $\lambda = \lambda_0 + \lambda_1$ $\bar{z} = \bar{z}_0 + \bar{z}'_1$

EQUILIBRIUM, $\lambda_0 = \text{CONSTANT}$
 $\frac{\partial \lambda_0}{\partial z} = 0$

LINEARIZING

$$\frac{\partial \lambda_1}{\partial s} - \lambda_0 \frac{\partial \bar{z}'_1}{\partial z} = 0 \tag{g2a}$$

$$\frac{\partial \bar{z}'_1}{\partial s} + \frac{c_s^2}{\lambda_0 v_0^2} \frac{\partial \lambda_1}{\partial z} = 0 \tag{g2b}$$

TAKING $\frac{\partial}{\partial s}$ of (g2a) & $\frac{\partial}{\partial z}$ of g2b and combining:

$$\Rightarrow \frac{\partial^2 \lambda_1}{\partial s^2} - \frac{c_s^2}{v_0^2} \frac{\partial^2 \lambda_1}{\partial z^2} = 0 \Rightarrow \text{WAVE EQUATION}$$

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50 SHEETS HALF SIZE
100 SHEETS QUARTER SIZE
100 SHEETS EIGHTH SIZE
200 SHEETS SIXTEENTH SIZE
200 SHEETS THIRTY-SECOND SIZE
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MADE IN U.S.A.
National Brand

SOLVING WAVE EQUATION

$$\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{c_s^2}{v_0^2} \frac{\partial^2 \lambda_1}{\partial z^2} = 0$$

Let $\lambda_1 = \tilde{\lambda}_1 \exp \left[\frac{i\omega}{v_0} s \pm ikz \right]$

$$-\frac{\omega^2}{v_0^2} + \frac{k^2 c_s^2}{v_0^2} = 0 \Rightarrow \omega = c_s k$$

\Rightarrow PHASE & GROUP VELOCITY OF WAVES = c_s
(in beam frame)

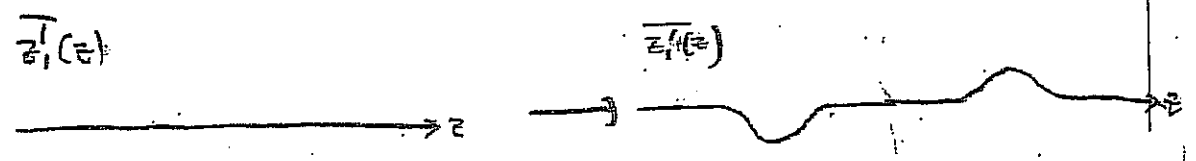
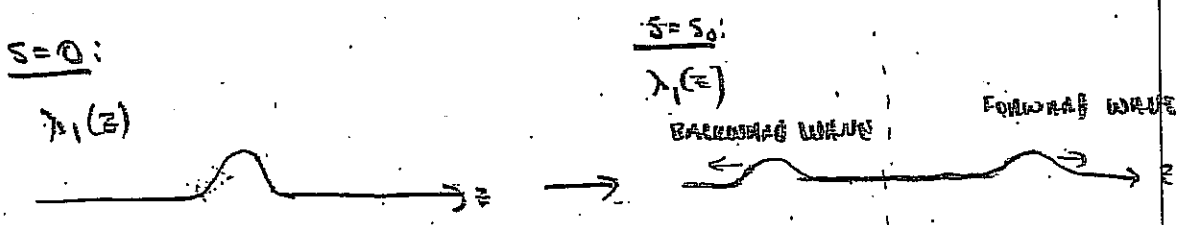
GENERAL SOLUTION

$$\lambda_1 = \lambda_0 f_+[u_+] + \lambda_0 f_-[u_-]$$

where $u_+ = z + \frac{c_s s}{v_0} + C_0$ & $u_- = z - \frac{c_s s}{v_0} + C_0$

& $f_+[u_+]$ & $f_-[u_-]$ are any functions of the argument & C_0 is an arbitrary constant.

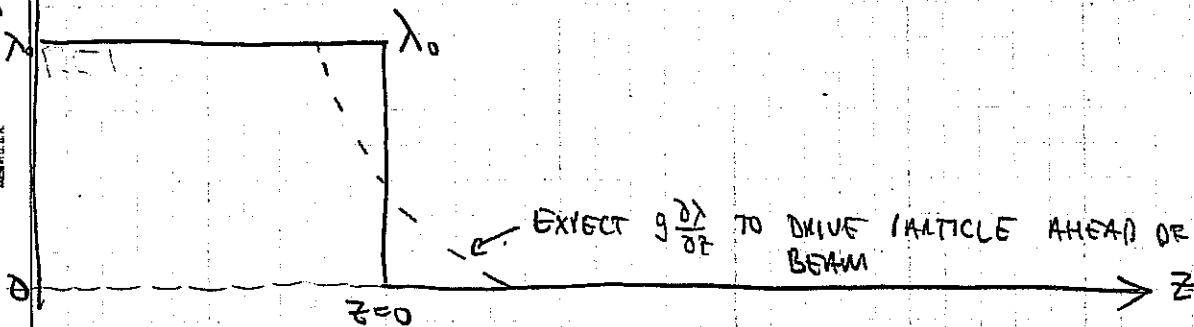
$$\tilde{z}'_1 = \frac{c_s}{v_0} [-f_+[u_+] + f_-[u_-]]$$



BEAM ENDS & RAREFACTION WAVES

(FALTINGS & LEE,
J. APPL. PHYS. 61, 5219)
(Also Landau & Lifshitz,
FLUID MECHANICS)

SUPPOSE YOU START WITH A PULSE THAT ENDS WITH A STEP FUNCTION IN λ . WHAT HAPPENS TO THE END?



TO ANALYZE: RETURN TO NON-LINEAR FLUID EQUATIONS (SINCE $\delta\lambda \sim \lambda_0$) (91):

$$\frac{\partial \lambda}{\partial s} + \lambda \frac{\partial z'}{\partial z} + z' \frac{\partial \lambda}{\partial z} = 0 \quad (\text{CONTINUITY})$$

$$\frac{\partial z'}{\partial s} + z' \frac{\partial z'}{\partial z} + \frac{c^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} = 0 \quad (\text{MOMENTUM})$$

1ST IT IS CONVENIENT TO DEFINE: $\Lambda \equiv \lambda / \lambda_0$

$$V \equiv \frac{v_0}{c} z'$$

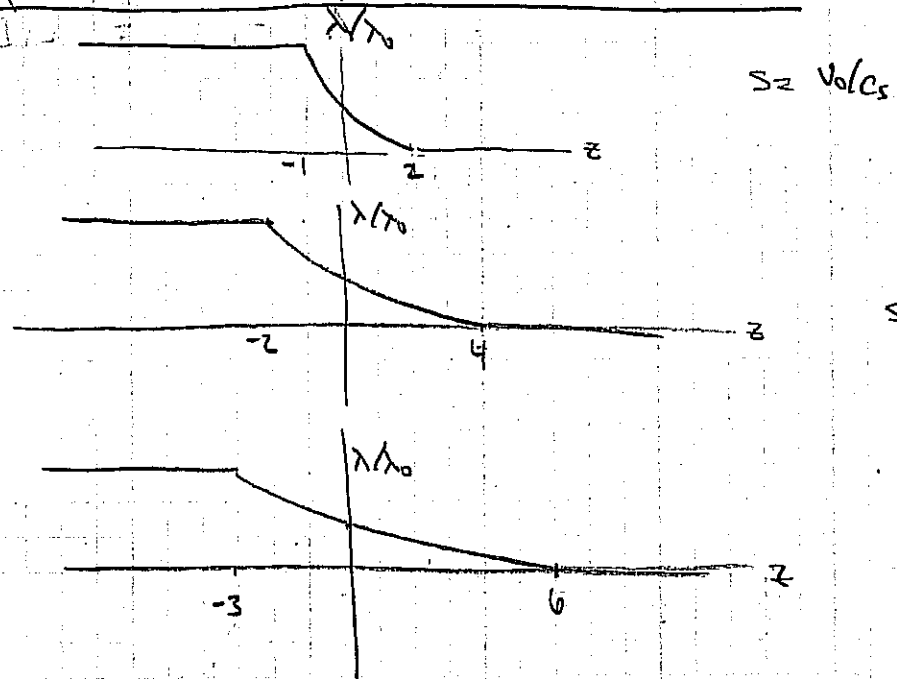
$$z \equiv \frac{v_0}{c} z$$

$$(c^2 \equiv \frac{g}{\omega} \frac{g \lambda_0}{4\pi \epsilon_0})$$

$$\Rightarrow \frac{\partial \Lambda}{\partial s} + \Lambda \frac{\partial V}{\partial z} + V \frac{\partial \Lambda}{\partial z} = 0 \quad (\text{CONTINUITY})$$

$$\frac{\partial V}{\partial s} + V \frac{\partial V}{\partial z} + \frac{\partial \Lambda}{\partial z} = 0 \quad (\text{MOMENTUM})$$

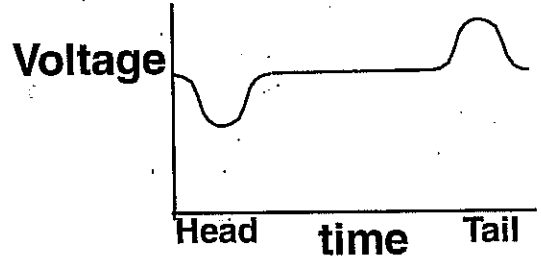
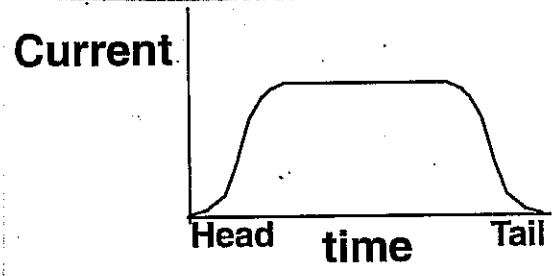
SNAPSHOTS OF λ/λ_0 VS z AT VARIOUS s



HOW DOES ONE PREVENT "END EROSION"?

APPLY EARL PULSES AT END OF BEAM:

$$V \sim E_z = \frac{1}{4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$$



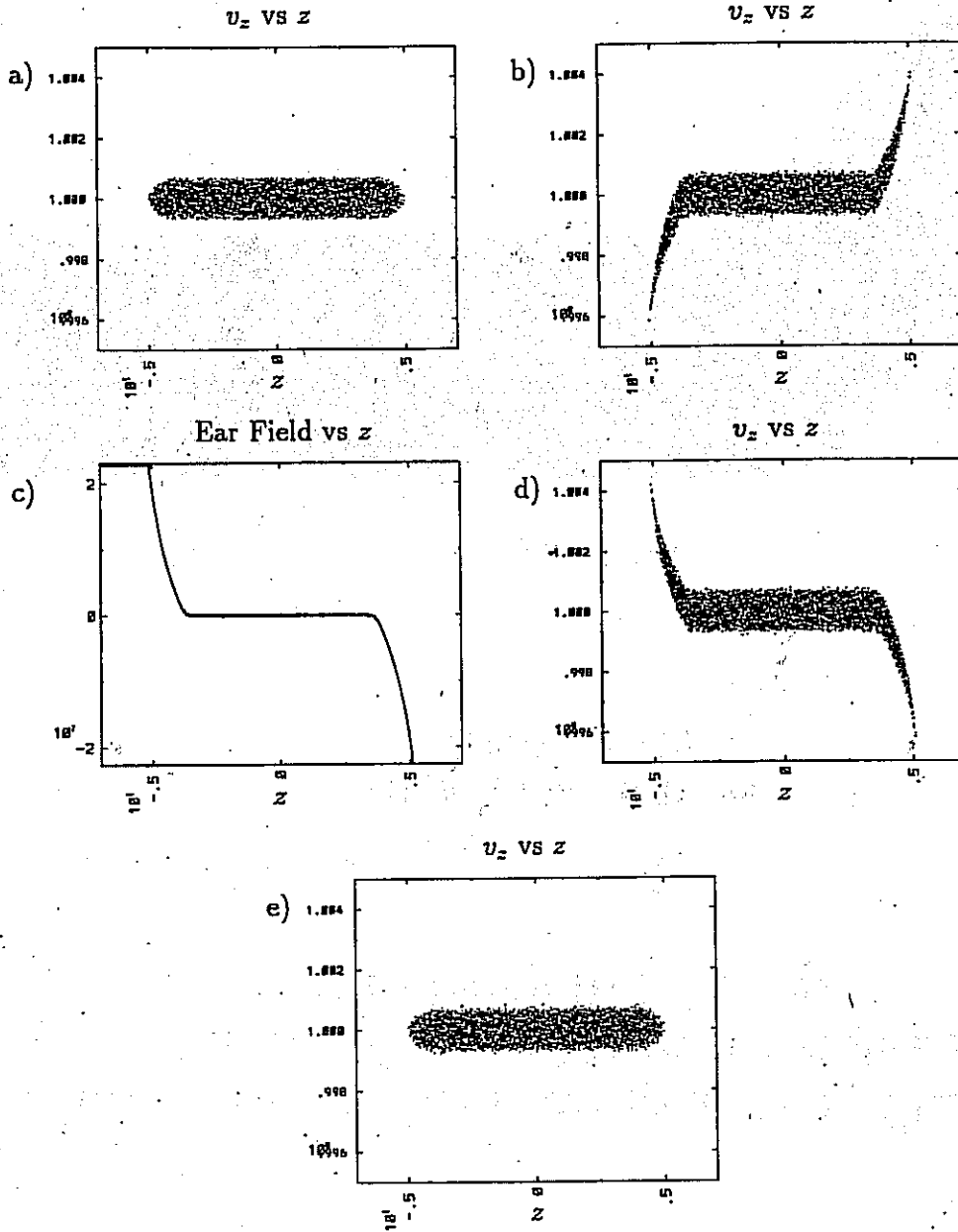


Figure 6.4: One cycle of intermittently-applied ears. (a) Initial phase space (b) Beam expands (c) Ear Field is applied (d) Beam is compressed (e) Beam expands back to its initial state

from D. Callahan Miller
PhD thesis, U.C. Davis, 1994.