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U.C. Berkeley

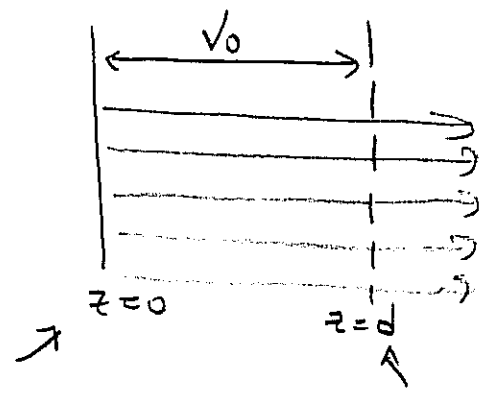
Injectors and longitudinal physics -- III

1. Longitudinal cooling from acceleration
2. Longitudinal instability
3. Bunch compression
4. Neuffer distribution

# LONGITUDINAL COOLING

1. DURING INJECTION BEAM UNDERGOES LARGE LONGITUDINAL EXPANSION
2.  $T_{L0} = T_{H0}$  AT SOURCE, BUT  $T_L \neq T_H$  AFTER ACCELERATION
3. IMPLICATIONS FOR BEAM STABILITY AND EMITTANCE EVOLUTION

CONSIDER 1D DIODE:



AT SOURCE

$$E_0 = \frac{p_{z0}^2}{2m}$$

$$\Delta E_{H0} \equiv \frac{\langle p_{z0}^2 \rangle}{2m} = \frac{1}{2} kT_{H0}$$

AT END OF DIODE

$$E_f = qV_0 + \frac{p_{zf}^2}{2m} = \frac{p_{zf}^2}{2m}$$

$$\Delta E_{Hf} = \Delta E_{H0} \neq \frac{1}{2} kT_f$$

SINCE  $E_H = \frac{p_z^2}{2m} \Rightarrow \Delta E_H = \frac{2p_z \Delta p_z}{2m}$

$$\frac{\Delta E}{E} = \frac{2 \Delta p_z}{p_z}$$

$$\frac{1}{2} kT_f \approx \frac{\Delta p_{zf}^2}{2m} = \left( \frac{p_{zf} \Delta E_f}{2E_f} \right)^2 \frac{1}{2m} = \frac{\Delta E_f^2}{4E_f} = \frac{kT_0}{2} \left[ \frac{1}{2} \frac{kT_0}{qV_0} \right]$$

$$\Rightarrow \boxed{kT_f = \frac{1}{2} kT_0 \left[ \frac{kT_0}{qV_0} \right]}$$

$$kT_f = \frac{1}{2} kT_0 \left[ \frac{kT_0}{qV_0} \right] \ll 1$$

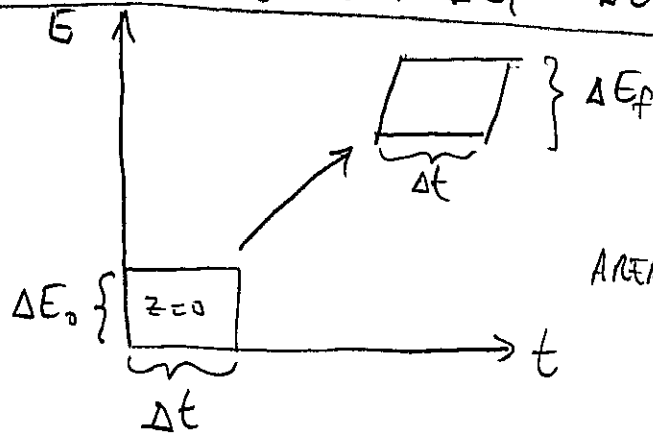
EXAMPLE  $1000^\circ\text{C} \Leftrightarrow 0.1 \text{ eV}$

FOR  $V_0 = 1 \text{ MeV}$

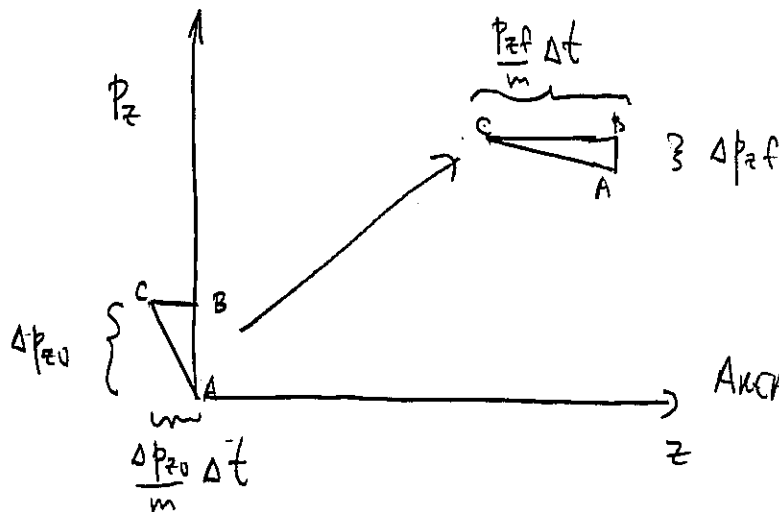
$kT_0 = 0.1 \text{ eV}$

$kT_f = 5 \times 10^{-9} \text{ eV}$

How CAN  $kT_f \ll kT_0$  BUT  $\Delta E_f = \Delta E_0$ ?



AREA IS CONSERVED  
(PULSE DURATION STAYS THE SAME.)



(BUNCH LENGTH GROWS)  
AREA IS CONSERVED

$$\frac{1}{2} \frac{\Delta p_{z0}^2 \Delta t}{m} = \frac{1}{2} \Delta p_{zf} \left( \frac{p_{zf}}{m} \right) \Delta t$$

$$\Rightarrow \Delta p_{zf} = \frac{\Delta p_{z0}^2}{p_{zf}}$$

$$\Rightarrow kT_f = \frac{1}{2} kT_0 \left( \frac{kT_0}{qV_0} \right)$$

# CHANGE IN NOTATION

4

NOTE:  $\bar{z}' \equiv \left\langle \frac{dz}{ds} \right\rangle$ ;  $s = v_0 t$

Let  $u = \left\langle \frac{dz}{dt} \right\rangle$ ; then  $u = v_0 \bar{z}'$   
 = fluid velocity in comoving frame

So

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z} (\lambda \bar{z}') = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} (\lambda u) = 0}$$

$$\neq \frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial}{\partial z} \bar{z}' + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda \Delta \bar{z}'^2) = \ddot{\bar{z}}$$

$$\Rightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda [\langle v_z^2 \rangle - u^2]) = \ddot{\bar{z}}$$

Since  $p_z = m \int_{-\infty}^{\infty} n [v_z^2 - u^2] dv_z$  where  $n = \frac{\lambda}{\pi v_b^2}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{\pi v_b^2}{m \lambda} \frac{\partial}{\partial z} p_z = \ddot{\bar{z}}}$$

where  $\ddot{\bar{z}} = \frac{d^2 \bar{z}}{dt^2}$   
 $= \frac{1}{v_0} \frac{d^2 z}{dt^2}$

# "LONGITUDINAL" or "RESISTIVE WALL" INSTABILITY

Let us return to the 1-D FLUID EQUATIONS

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda u = 0$$

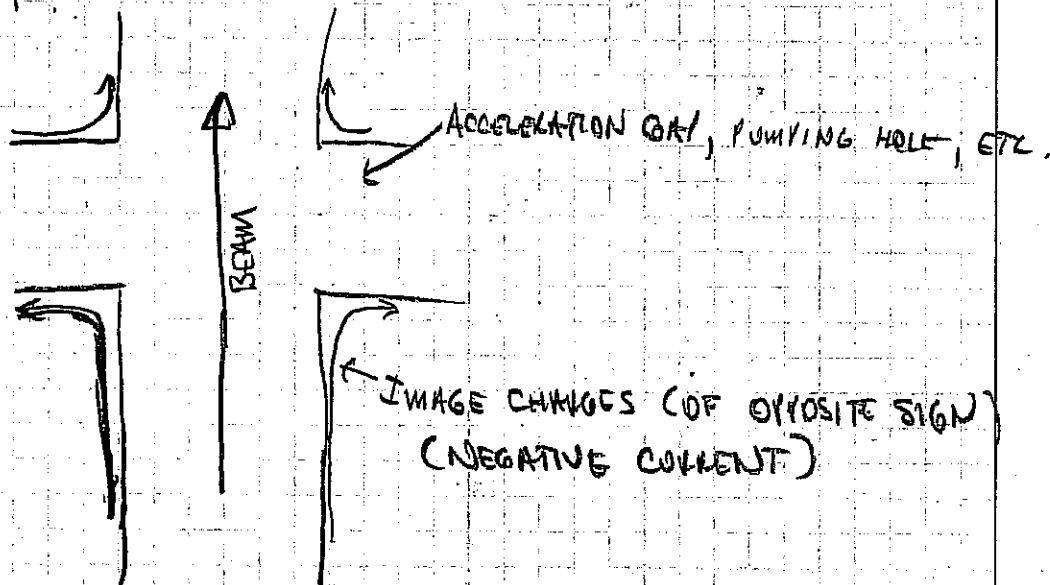
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p_e}{\partial z} = \frac{-g \rho}{4\pi \epsilon_0 m v_0^2} \frac{\partial \lambda}{\partial z} + \frac{\rho E_z}{m}$$

↑  
IGNORE  
AGAIN

↑  
EXTERNALLY  
GENERATED

SEE  
REISER 6.3.2.  
CALLAHAN-MILLER, PH  
D. DISSERTATION,  
U.C. DAVIS, 1994

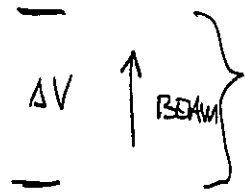
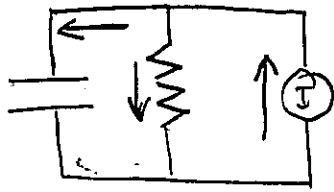
AS BEAM PASSES CONDUCTING SURFACE IMAGE CHARGE AND CURRENT INTERACTS WITH BEAM. HIGHLY GEOMETRY DEPENDENT.



CAN BE CALCULATED APPROXIMATELY USING CIRCUIT MODEL.

RESISTIVITY IN WALL, AND COMPLICATED ELECTRON FLOW PATTERNS CREATE A RETARDING ELECTRIC FIELD ON BEAM.

# MODEL OF IMPEDANCE (IN LONG WAVELENGTH REGIME)



ONE MODULE OF MANY, EACH SEPARATED BY DISTANCE L

$$I = C \frac{d\Delta V}{dt} + \frac{\Delta V}{R}$$

$$I = [CL] \frac{d\Delta V/L}{dt} + \frac{\Delta V/L}{R/L}$$

$$E = -\frac{\Delta V}{L}$$

$$C^+ = CL$$

$$R^* = \frac{R}{L}$$

$$\text{LET } I = I_0 + I_1 e^{-i\omega t}$$

$$E = E_0 + E_1 e^{-i\omega t}$$

$$I_1 = i\omega C^+ E_1 - \frac{E_1}{R^*}$$

$$\Rightarrow E_1 = \frac{-R^*}{1 - i\omega C^+ R^*} I_1$$

$$Z^* \equiv \frac{-E_1}{I_1} = \frac{R^*}{1 - i\omega C^+ R^*}$$

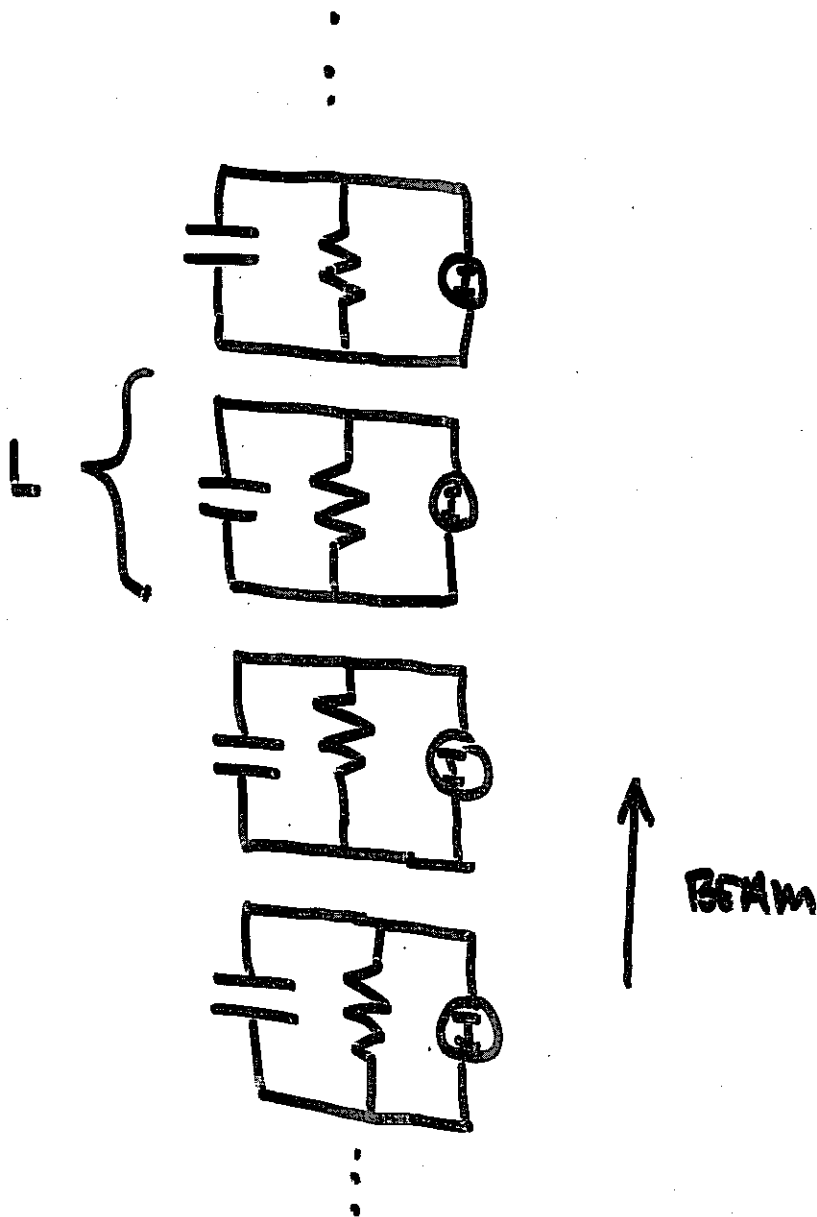
## RETURNING TO THE 1D FLUID EQUATIONS

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = \frac{-\rho g}{4\pi\epsilon_0 m} \frac{\partial \lambda}{\partial z} + \frac{\rho E_0}{m}$$

$$\text{Let } \lambda = \lambda_0 + \lambda_1 \exp[i(kz - \omega t)]$$

$$u = v_0 + u_1 \exp[i(kz - \omega t)]$$



CONTINUOUS LIMIT:

$$R^* = R/L$$

$$C^* = CL$$

$$E = \frac{\Delta V}{L}$$

Resistance per unit length  
 $C^*$  per unit length

AVERAGE ELECTRIC  
 FIELD

$$-i\omega\lambda_1 + ik\lambda_0 u_1 + ikv_0\lambda_1 = 0$$

$$-i\omega u_1 + ikv_0 u_1 + \underbrace{\frac{ikq\lambda_1}{4\pi\epsilon_0 m}}_{= \frac{ikc_s^2}{\lambda_0} \lambda_1} + \frac{q}{m} z^* (\lambda_0 v_0 + v_0 \lambda_1) = 0$$
  
$$\underbrace{\hspace{10em}}_{= I_1}$$

$$\begin{bmatrix} \omega - kv_0 & -k\lambda_0 \\ -\frac{c_s^2 k}{\lambda_0} + \frac{iq}{m} z^* v_0 & \omega - kv_0 + \frac{iq}{m} z^* \lambda_0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ u_1 \end{bmatrix} = 0$$

THE DETERMINANT OF THE ABOVE MATRIX MUST VANISH:

$$(\omega - kv_0)^2 + \frac{iq}{m} z^* \lambda_0 (\omega - kv_0) - c_s^2 k^2 + \frac{iq}{m} z^* \lambda_0 v_0 k = 0$$

$$\boxed{(\omega - kv_0)^2 - c_s^2 k^2 + \frac{iq z^* \lambda_0 \omega}{m} = 0} \quad (\text{LAB FRAMES})$$

Using a Galilean transformation, in the beam frame:

$$\omega' = \omega - kv_0$$
$$k' = k$$

' denotes beam frame

$$\boxed{\omega'^2 - c_s^2 k'^2 + \frac{iq z^*(\omega') \lambda_0 (\omega' + k'v_0)}{m} = 0} \quad (\text{BEAM FRAMES})$$

NOTE  $z^*(\omega') = z^*(\omega = \omega' + k'v_0)$



CASE I PURE RESISTIVE IMPEDANCE  $Z^* = R^*$  (REAL)

$$\omega' z = \pm c_s k' \sqrt{1 - i R^* \frac{g \lambda_0}{m c_s^2 k'^2} (\omega' + k' v_0)}$$

Using  $c_s^2 = \frac{g \lambda_0}{4 \pi \epsilon_0 m}$  and  $\frac{\omega'}{k'} \sim c_s \ll v_0$

$$\omega' = \pm c_s k' \sqrt{1 - i R^* \left(\frac{4 \pi \epsilon_0}{g}\right) \frac{v_0}{k'}}$$

$$\approx \pm \left[ c_s k' - i \frac{c_s v_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* \right]$$

Since  $\lambda_1, E_1 \sim \exp [i(k' z' - \omega' t')]$

Choosing "+" ( $\text{Re } \omega' > 0$ )  $\Rightarrow z' = c_s t'$  line of const phase  $\Rightarrow$  Forward propagation

( $\text{Im } \omega' < 0$ )  $\Rightarrow \lambda_1 \sim \exp \left[ -\frac{c_s v_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* t' \right] \Rightarrow$  DECAYING PERTURBATION

CHOOSING "-"

( $\text{Re } \omega' < 0$ )  $\Rightarrow z' = -c_s t'$  is line of constant phase

$\Rightarrow$  BACKWARD PROPAGATING

$$\Rightarrow \lambda_1 \sim \exp \left[ \underbrace{\frac{c_s v_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* t'}_G \right]$$

INSTABILITY!

$$\lambda_1 \approx \lambda_{10} \exp[G]$$

(9)

$$G = \left[ \frac{C_s V_0}{2} \left( \frac{4\pi\epsilon_0}{g} \right) R^2 t \right] = \begin{matrix} \text{LOGARITHMIC} \\ \text{GAIN} \\ \text{OF} \\ \text{INSTABILITY} \end{matrix} = \ln \left( \frac{\lambda_{\text{final}}}{\lambda_{\text{initial}}} \right)$$

Now  $t_{\text{max}} = \left\{ \begin{matrix} \text{min} \\ \left\{ \begin{matrix} l_b / c_s \\ t_{\text{residence}} \end{matrix} \right. \end{matrix} \right.$

TRANSIT TIME FOR PERTURBATION TO TRAVEL FROM HEAD TO TAIL

RESIDENCE TIME WITHIN ACCELERATOR

IF upper condition holds

$$G \sim \frac{V_0^2}{2} \left( \frac{4\pi\epsilon_0}{g} \right) R^2 t$$

IF lower condition holds

$$G \sim \sqrt{\lambda}$$

$$\epsilon = QV$$

(10)

$$I \sim \frac{6 \text{ MJ}}{4 \text{ GeV} \cdot 200 \text{ ns}} \sim \frac{QV}{V \Delta t} \\ \sim 7.5 \text{ kA}$$

EXAMPLE:

FOR MATCHED BEAM IMPEDANCE

$$R^* = \frac{\Delta V / \Delta s}{I} \sim \frac{10^6 \text{ V/m}}{10 \text{ kA}} \sim 100 \text{ } \Omega / \text{m}$$

$$v_0 \sim 0.2c$$

$$\Delta t \sim 200 \text{ ns}$$

$$G \sim \frac{v_0^2}{2} \left( \frac{4\pi\epsilon_0}{g} \right) R^* \Delta t$$

$$\sim 3.6$$

( AN EARLY CONCERN  
FOR HEAVY ION FUSION )

$$R^* = 100 \Omega/m$$

(1)

FOR ALL  
SIMULATIONS  
(p 11-15)

$$V_0 = c/3$$

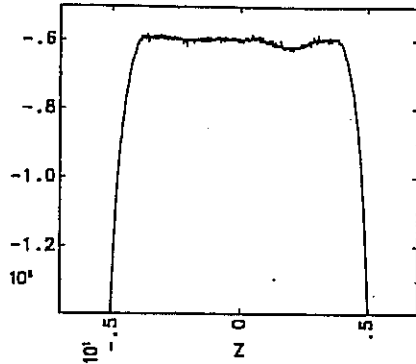
$$I = 3 \text{ kA}$$

$$l_b = 10 \text{ m}$$

$$\frac{v_b}{v_p} = 0.4$$

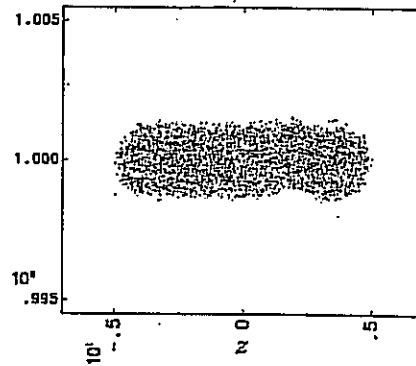
$$kT_{\perp} = kT_{\parallel} = 10 \text{ keV}$$

Electrostatic Potential on Axis vs z

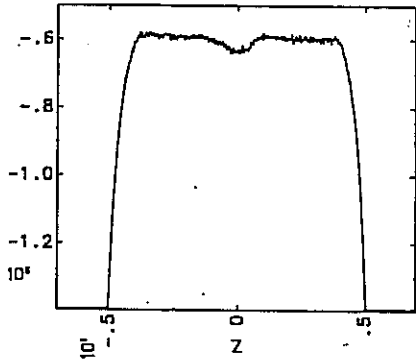


a)

$v_z$  vs z

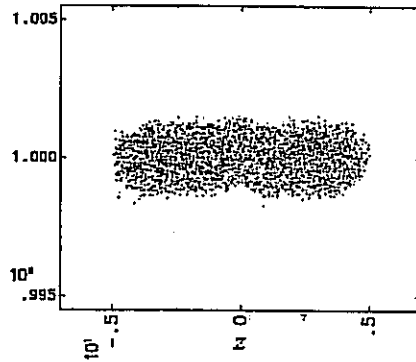


Electrostatic Potential on Axis vs z

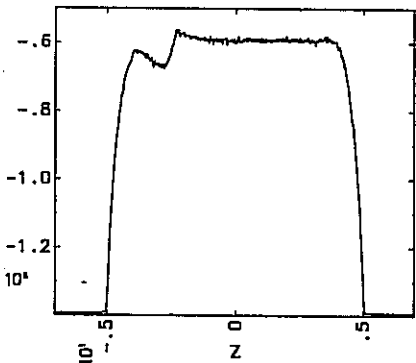


b)

$v_z$  vs z



Electrostatic Potential on Axis vs z



c)

$v_z$  vs z

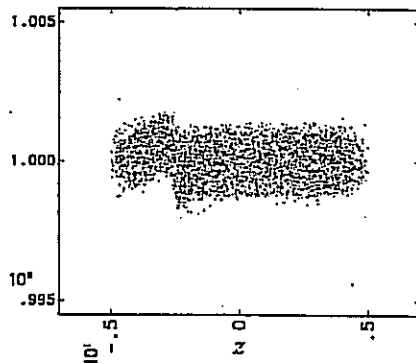


Figure 4.2: A simulation with  $100 \Omega/m$  resistance shows moderate growth. (a)  $6.6 \mu\text{s}$ , (b)  $10.9 \mu\text{s}$ , (c)  $17.5 \mu\text{s}$

from D.A. Callahan Miller, Ph.D. Thesis  
U.C. Davis, 1994

$$R^* = 100 \Omega/m$$

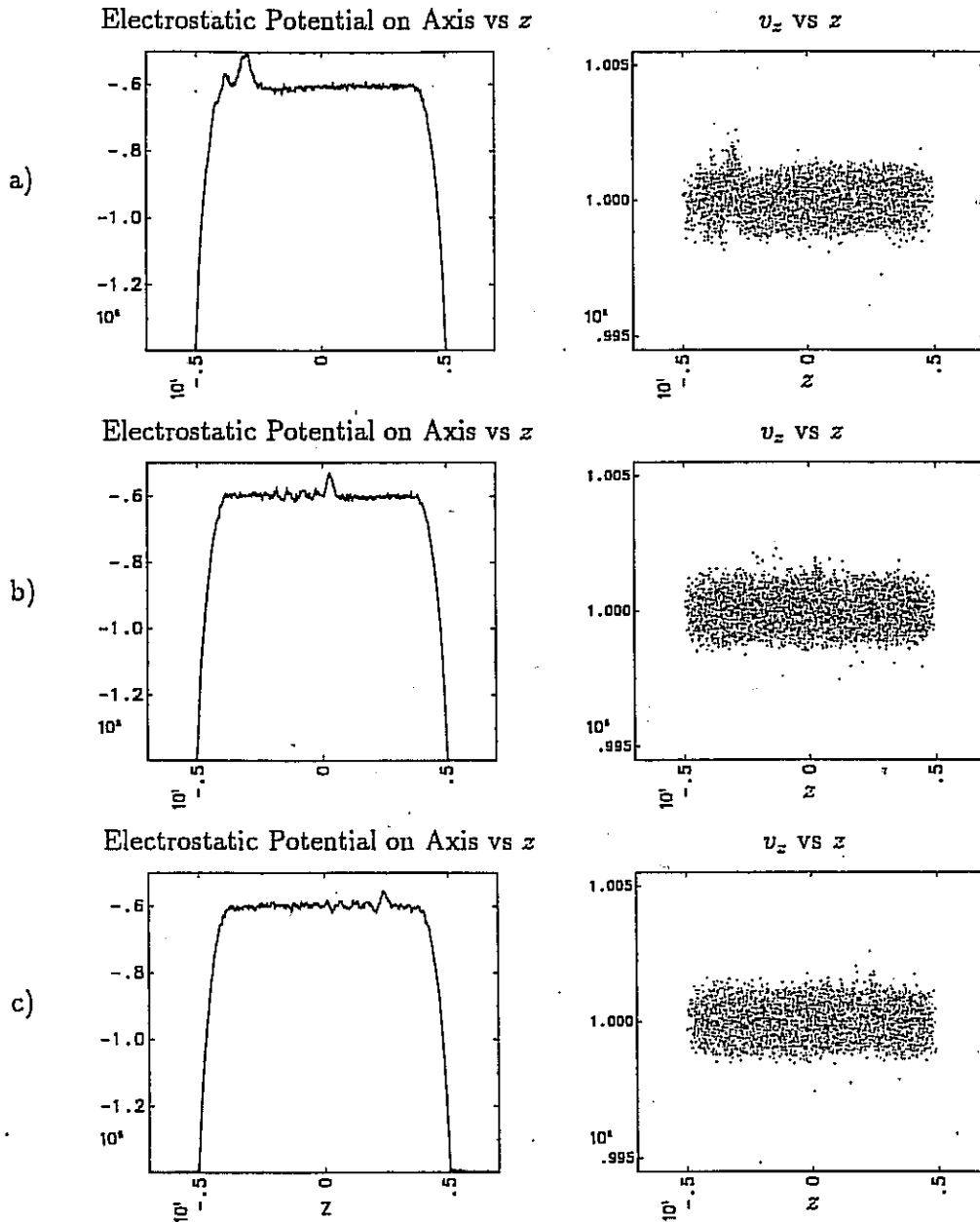


Figure 4.3: The perturbation reflects off the beam end and decays as it travels forward. (a) 28.4  $\mu$ s, (b) 35.0  $\mu$ s, (c) 39.4  $\mu$ s

from D.A. Callahan Miller, Ph.D. Thesis  
U.C. Davis, 1994  
(FORWARD WAVE)

$$R^* = 200 \Omega/m$$

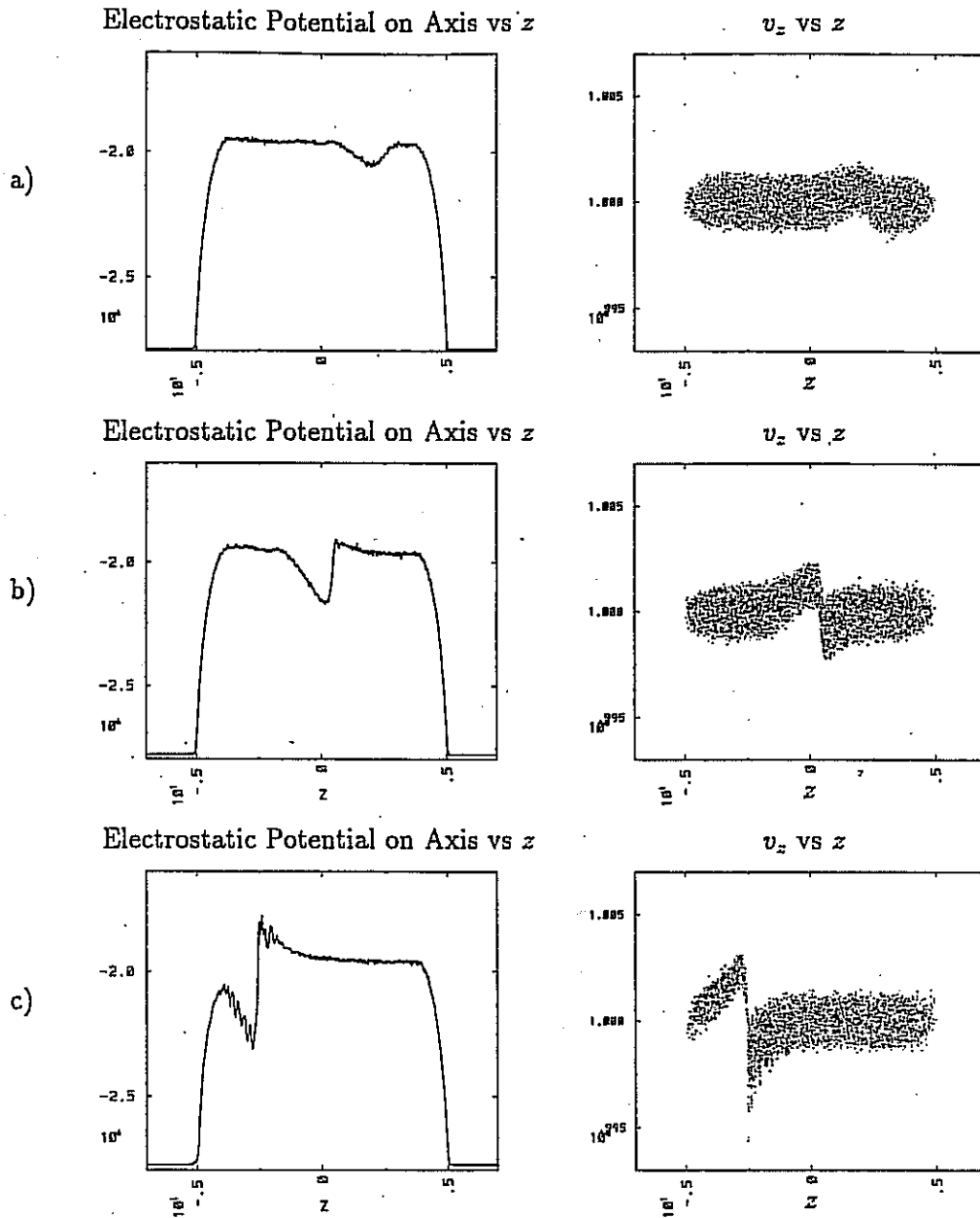


Figure 4.1: A simulation with 200 Ω/m resistance shows large amounts of growth. (a) 6.6 μs, (b) 10.9 μs, (c) 17.5 μs

From D.A. Callahan Miller, Ph.D. Thesis  
U.C. Davis, 1994

## CASE II RESISTIVE + CAPACITIVE IMPEDANCE

$$Z^* = \frac{R^*}{1 - i\omega C^+ R^*} = \frac{R^* + i\omega C^+ R^{*2}}{1 + \omega^2 C^{+2} R^{*2}}$$

GOING BACK TO (A07-7):

IN LAB FRAME:

$$(\omega - kv_0)^2 - c_s^2 k^2 + \frac{i q R^* \lambda_0 \omega}{m(1 + \omega^2 C^{+2} R^{*2})} - \frac{q \omega^2 C^+ R^{*2} \lambda_0}{m(1 + \omega^2 C^{+2} R^{*2})} = 0$$

$$(\omega - kv_0)^2 - c_s^2 k^2 - \frac{4\pi\epsilon_0 \omega^2 C^+ R^{*2} c_s^2}{g(1 + \omega^2 C^{+2} R^{*2})} + \frac{i 4\pi\epsilon_0 c_s^2 R_x^* \omega}{g(1 + \omega^2 C^{+2} R^{*2})}$$

IN BEAM FRAME:

$$\omega'^2 - c_s^2 k^2 - \frac{4\pi\epsilon_0 (\omega' + kv_0)^2 C^+ R^{*2} c_s^2}{g(1 + (\omega' + kv_0)^2 C^{+2} R^{*2})} + \frac{i 4\pi\epsilon_0 c_s^2 R_x^* (\omega' + kv_0)}{g(1 + \omega'^2 C^{+2} R^{*2})}$$

So if one takes limit  $C \rightarrow \infty$  the final two terms tend to zero. Thus capacitance has reduced the instability growth rate.

$$\begin{aligned} RC^* &= 2 \times 10^{-8} \text{ s} \\ R^* &= 100 \Omega/\text{m} \end{aligned}$$

15

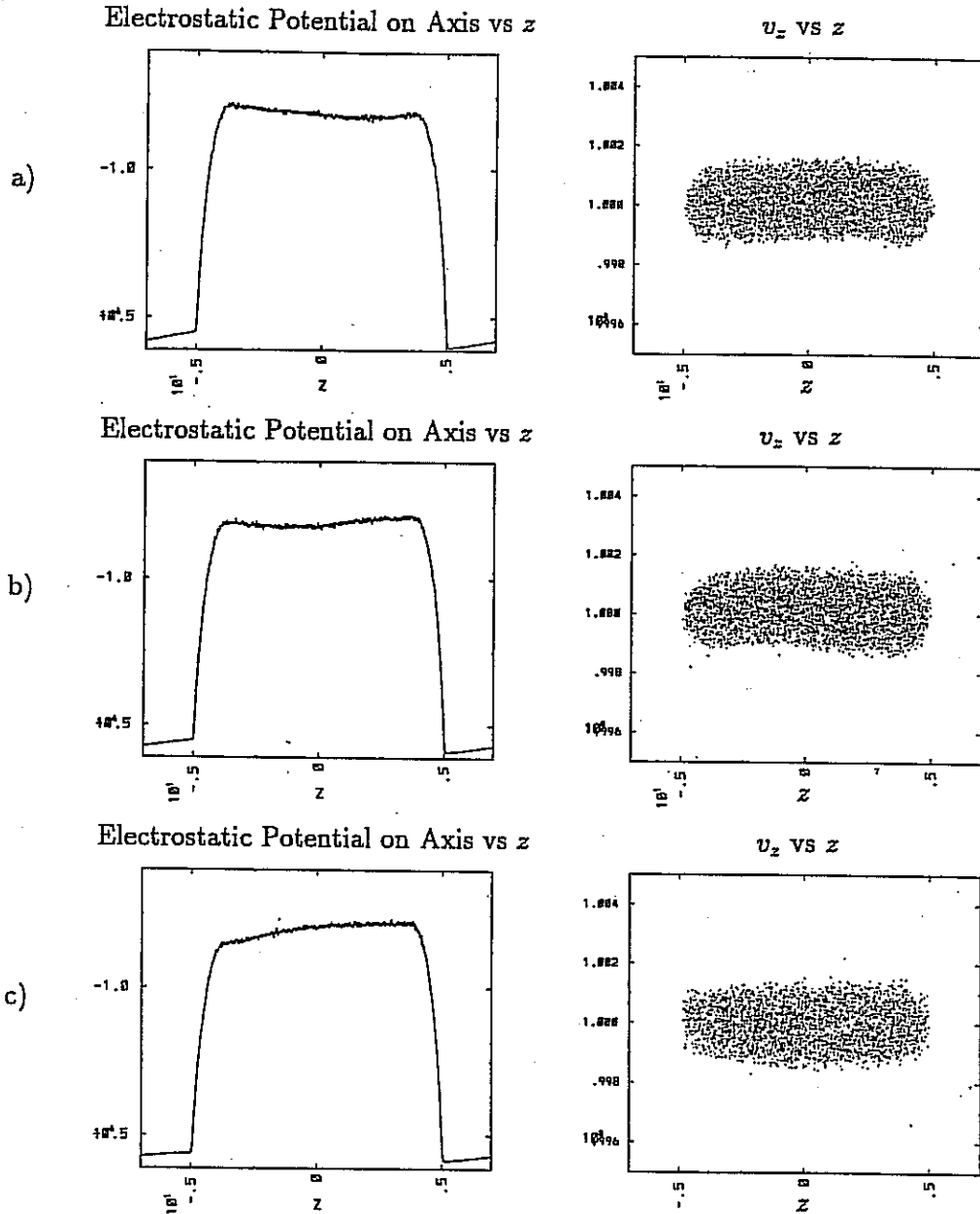


Figure 4.6: When capacitance is added to the system, a larger perturbation is launched, but little growth occurs (a)  $6.6 \mu\text{s}$ , (b)  $10.9 \mu\text{s}$ , (c)  $17.5 \mu\text{s}$

from D.A. Callahan Miller, Ph. D. Thesis,  
U.C. Davis, 1994



# SUMMARY OF LONGITUDINAL INSTABILITY

"RESISTIVE WALL" OR "LONGITUDINAL" INSTABILITY HAS POTENTIAL TO DEGRADE LONGITUDINAL EMITTANCE IN HIGH CURRENT ACCELERATORS.

HOWEVER, CAVACITANCE (e.g. FROM ACCELERATING GUNS) DECREASES GROWTH CAN MITIGATE INSTABILITY,

## NOT DISCUSSED:

1. LONGITUDINAL TEMPERATURE DAMAGE INSTABILITY (e.g. REISER 6.3.3)
2. FEED BACK HAS BEEN PROPOSED TO CONTROL INSTABILITY IF NEEDED

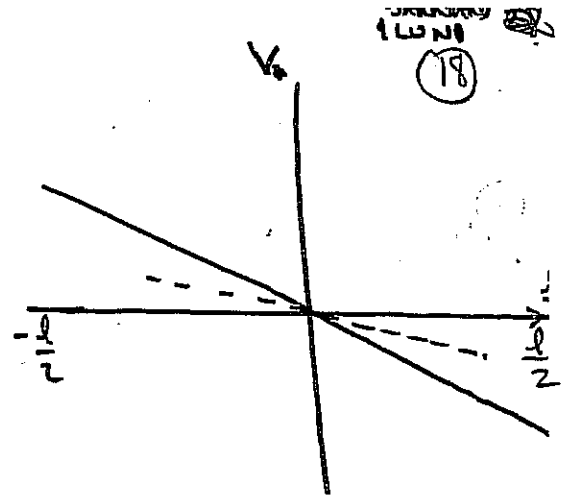
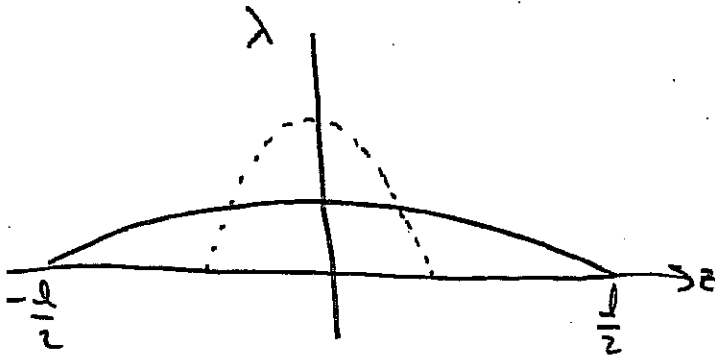
## DRIFT COMPRESSION

OBJECTS :

APPLY A HEAD-TO-TAIL VELOCITY TILT TO  
 INCREASE CURRENT BY DECREASING PULSE DURATION

DURING COMPRESSION "TAILS" ARE NOT REQUIRED

AT END OF DRIFT COMPRESSION, VELOCITY "TILT"  
 SHOULD BE MINIMIZED, SO THAT CHROMATIC  
 ABERRATIONS IN FINAL FOCUS ARE MINIMIZED.



$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

CONTINUITY EQUATION

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = -\frac{qg}{m 4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$$

MOMENTUM EQUATION

$$\text{LET } \lambda = \lambda_0(t) \left(1 - \frac{4z^2}{l^2(t)}\right)$$

← PARABOLIC LINE CHARGE PROFILE

$$v = -\Delta v(t) \frac{z}{l(t)}$$

← LINEAR VELOCITY PROFILE

① MASS conservation:

$$Q_c = \int_{-l/2}^{l/2} \lambda dz = \lambda_0 \int_{-l/2}^{l/2} \left(1 - \frac{4z^2}{l^2}\right) dz = \frac{2}{3} \lambda_0 l = \text{constant}$$

(but

$$\lambda_0 = \lambda_0(t)$$

$$\& l = l(t))$$

# CALCULATING PARTIAL DERIVATIVES:

CALCULATED  
BY  
19

$$\frac{\partial \lambda}{\partial t} = \dot{\lambda}_0 \left(1 - \frac{4z^2}{l^2}\right) + 2\lambda_0 \left(\frac{4z}{l^2}\right) \dot{z}$$

$$\frac{\partial \lambda}{\partial z} = -\frac{8z}{l^2} \lambda_0$$

$$\frac{\partial V}{\partial t} = -\dot{\Delta V} \left(\frac{z}{l}\right) + \frac{\Delta V}{l^2} z \dot{z}$$

$$\frac{\partial V}{\partial z} = -\frac{\Delta V}{l}$$

FROM DEFINITION OF  
 $\Delta V$  &  $\dot{z}$ :  
 $\Delta V = -\dot{z}$

② CONTINUITY EQUATION  $\Rightarrow \left(1 - \frac{4z^2}{l^2}\right) \left(\dot{\lambda}_0 - \frac{\Delta V \lambda_0}{l}\right) = 0$

③ MOMENTUM EQUATION  $\Rightarrow \left(\frac{z}{l}\right) \left[-\dot{\Delta V} + \frac{\dot{z} \Delta V}{l} + \frac{\Delta V^2}{l} + \frac{8z \dot{z}}{4\pi \epsilon_0 l} \lambda_0\right] = 0$

① & ②  $\Rightarrow \frac{\dot{\lambda}_0}{\lambda_0} = \frac{\Delta V}{l} = -\frac{\dot{z}}{l}$  ④

③ & ④  $\Rightarrow \ddot{z} - \frac{12z \dot{z}}{4\pi \epsilon_0 l} \frac{Q_c}{l^2} = 0$

where  $Q_c = \frac{2}{3} \lambda_0 l = \text{const.}$   
III  
CHANGE  
IN  
MOMENTUM (NOT  
REVERSE)

LONGITUDINAL "ENVELOPE" EQUATION  
(WITHOUT EMITTANCE)

MULTIPLY BY  $\dot{l}$  & INTEGRATE:

$$\frac{\dot{l}^2}{2} + \frac{1299}{4\pi\epsilon_0 m} \frac{Q_c}{l} = \frac{\dot{l}_f^2}{2} + \frac{1299}{4\pi\epsilon_0 m} \frac{Q_c}{l_f}$$

HERE SUBSCRIPT "f"  
= "final"

& SUBSCRIPT "o"  
= original or initial

$$\Rightarrow \dot{l}_o = \sqrt{\frac{1699}{4\pi\epsilon_0 m} \lambda_f \left[ 1 - \frac{l_f}{l_o} \right]}$$

Now  $Q_f = \frac{\lambda_f}{4\pi\epsilon_0 V_f}$  = FINAL PERVEANCE AT CENTER OF PARABOLIC TUBE

(NOTE  $Q_c = \frac{2}{3} \lambda_o l$  = CHARGE

WHEREAS  $Q_f$  = PERVEANCE (DIMENSIONLESS)

$C =$  COMPRESSION RATIO =  $\frac{l_o}{l_f}$

$\frac{\Delta V}{V_o} =$  velocity tilt =  $\frac{|\dot{l}|}{V_o}$

$$\rightarrow \frac{\Delta V}{V} = \sqrt{89 Q_f \left[ 1 - \frac{1}{C} \right]}$$

for  $Q_f = 10^{-4}$   
 $g = 1.1$   
 $C = 20$

$$\Rightarrow \frac{\Delta V}{V} = 0.029$$

$$\text{DRIFT LENGTH} \approx \frac{l}{\Delta V} V_o = \frac{l}{\Delta V/V} = 345 \text{ m for } l = 10 \text{ m}$$

Vlasov - equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial f}{\partial z'} = 0$$

Let  $\tilde{f}(z, z', s) \equiv \iiint f dx dx' dy dy'$

INTEGRATING VLASOV EQUATION:

If  $z'' \neq f(x, x', y, y')$ :

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \iiint x' \frac{\partial f}{\partial x} dx dx' dy dy' + \dots + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0} \quad \text{1 D Vlasov}$$

Now let  $\lambda \equiv q \int \tilde{f} dz'$ ;  $\lambda \bar{z}' = \int \tilde{f} z' dz'$ ;  $\lambda \bar{z}'^2 = \int \tilde{f} z'^2 dz'$

Also, let  $\Delta z'^2 \equiv \bar{z}'^2 - (\bar{z}')^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER  $z'$ :

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial (\lambda \bar{z}')}{\partial z} = 0} \quad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY  $z'$  & INTEGRATING VLASOV OVER  $z'$ :

$$\frac{\partial}{\partial s} \lambda \bar{z}' + \frac{\partial}{\partial z} \lambda \bar{z}'^2 - \lambda z'' = 0$$

DIVIDING BY  $\lambda$ , USING CONTINUITY EQUATION & DEFINITION OF  $\Delta z'^2$ :

$$\boxed{\underbrace{\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial}{\partial z} \bar{z}'}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda} \frac{\partial (\lambda \Delta z'^2)}{\partial z}}_{\text{PRESSURE TERM}} = \underbrace{\bar{z}''}_{\text{FORCE}}} \quad \text{(MOMENTUM EQUATION)}$$

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LONGITUDINAL ENVELOPE EQUATION

$$\frac{\partial^2 \tilde{\rho}}{\partial s^2} + z' \frac{\partial \tilde{\rho}}{\partial z} + z'' \frac{\partial \tilde{\rho}}{\partial z'} = 0$$

$Q_c = \text{total charge in bunch}$

$$\text{IF } z'' = -K(s)z + \frac{gg}{4\pi\epsilon_0 m v^2} \left( \frac{12 Q_c}{L^3} \right) z$$

$$\Rightarrow \frac{\partial}{\partial s} \langle z^2 \rangle = 2 \langle z z' \rangle$$

$$\frac{\partial}{\partial s} \langle z z' \rangle = \langle z'' z' \rangle + \frac{gg}{4\pi\epsilon_0 m v^2} \left( \frac{12 Q_c}{L^3} \right) \langle z z' \rangle - K(s) \langle z^2 \rangle$$

$$\frac{\partial}{\partial s} \langle z'^2 \rangle = 2 \left( \frac{gg}{4\pi\epsilon_0 m v^2} \right) \left( \frac{12 Q_c}{L^3} \right) \langle z z' \rangle - 2 K(s) \langle z z' \rangle$$

NOTE  $\langle z^2 \rangle = \frac{1}{Q_c} \int_{-L}^L \int_{-L/2}^{L/2} z^2 f(z, z') dz dz' = \frac{1}{20} L^2$

$$\epsilon_z^2 = 2s [ \langle z'^2 \rangle \langle z^2 \rangle - \langle z z' \rangle^2 ]$$

$$\Rightarrow \frac{d^2 L}{ds^2} = \frac{16 \epsilon_z^2}{L^3} + \frac{12 gg Q_c}{4\pi\epsilon_0 m v^2 L^2} - K(s)L$$

Let  $v_z = L/2$

$$\Rightarrow \frac{d^2 v_z}{ds^2} = \frac{\epsilon_z^2}{v_z^3} + \frac{3}{2} \frac{gg Q_c}{4\pi\epsilon_0 m v^2} \frac{1}{v_z^2} - K(s)v_z$$





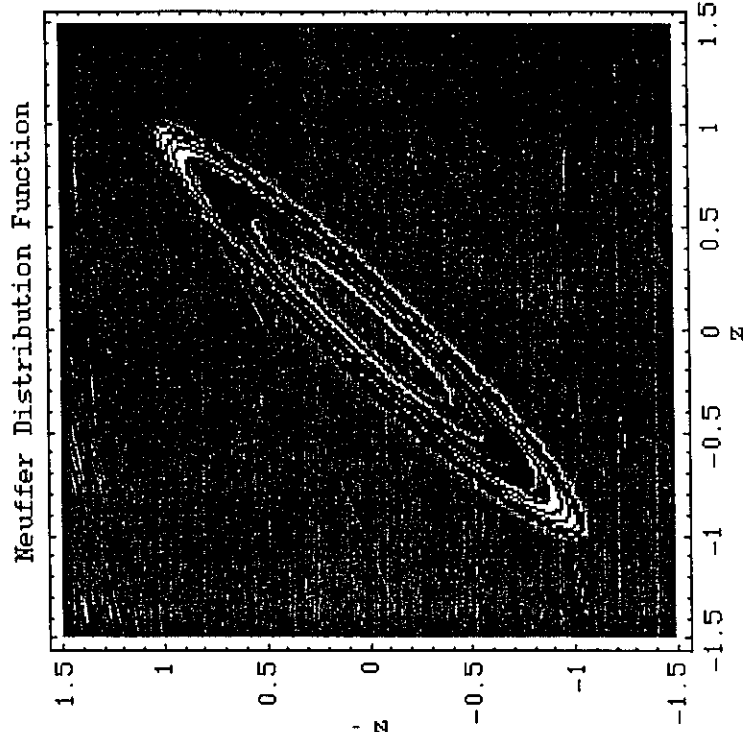
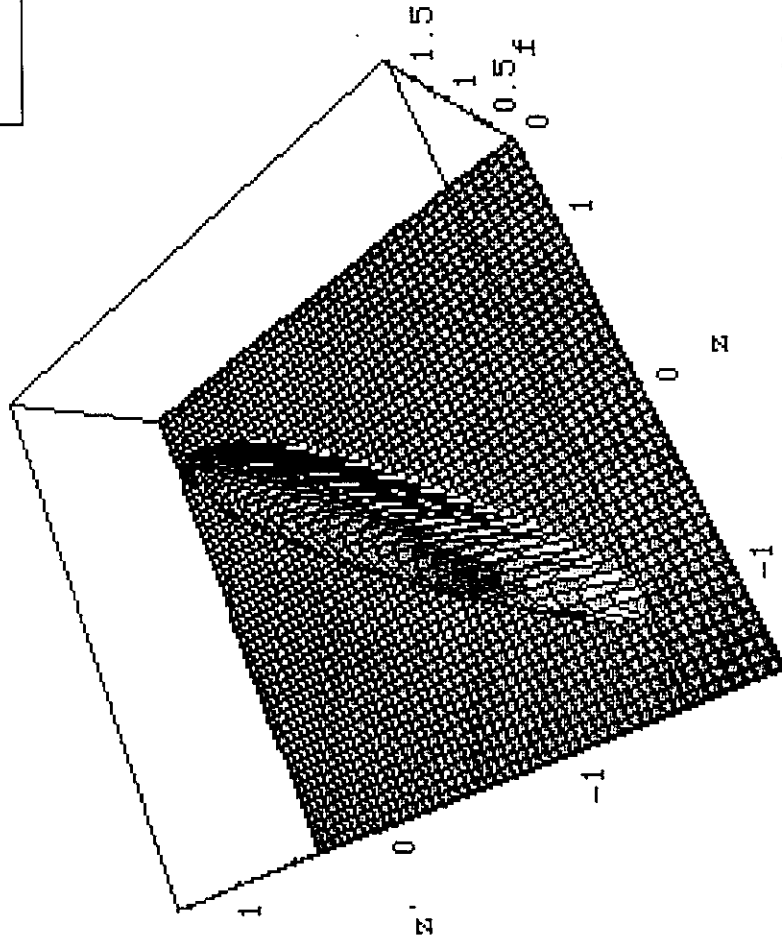
# Neuffer Distribution Function

$$f[z, z'] = \frac{3N}{2\pi\epsilon_z} \sqrt{1 - \frac{z^2}{r_z^2} - \frac{r_z^2 (z' - r_z'z/r_z)^2}{\epsilon_z^2}}$$

for:

$$-r_z \leq z \leq r_z$$

$$\frac{r_z'z}{r_z} - \frac{\epsilon_z}{r_z} \sqrt{1 - \frac{z^2}{r_z^2}} \leq z' \leq \frac{r_z'z}{r_z} + \frac{\epsilon_z}{r_z} \sqrt{1 - \frac{z^2}{r_z^2}}$$



Here  $N=r_z r_z' = 1$ ;  $\epsilon_z = 0.3$



NOTE:

- DISTRIBUTION FUNCTION HAS ELLIPTICAL BOUNDARY IN  $z-z'$  PHASE SPACE
- $\pi E_z$  IS AREA OF ELLIPSE AND  $K$  CONSTANT
- ANALOGOUS TO K-V DISTRIBUTION WITH LINEAR SPACE CHARGE FORCE AND SECOND ORDER ENVELOPE EQUATION TO DESCRIBE THE MOTION OF THE DISTRIBUTION:

$$\frac{d^2 r_z}{ds^2} = \frac{E_z^2}{r_z^3} + \frac{3}{2} \frac{AN}{r_z^2} - K(s) r_z$$

NOTE ALSO THAT NEUFFER FUNCTION CAN BE USED FOR BUNCHED BEAMS IN WHICH  $E_z \propto z$ , AS IN A UNIFORM DENSITY ELLIPSOID.

### Summary

#### 1D VLASOV EQUATION

g-factor model

$$\frac{\partial f^N}{\partial s} + z' \frac{\partial f^N}{\partial z} + z'' \frac{\partial f^N}{\partial z'} = 0$$

$$z'' = -g \frac{\partial \lambda}{4\pi \epsilon_0 m v_z^2 \partial z}$$

Leads to fluid equations:

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z} (\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda z'^2) + \frac{c_s^2}{\lambda_0 v_z^2} \frac{\partial \lambda}{\partial z} = 0$$

⇒ SPACE CHARGE WAVES

↳ LONGITUDINAL OR RESISTIVE WAVE INSTABILITY

⇒ SPACE CHARGE LATERALIZATION WAVES

⇒ PARABOLIC BUNCH COMPRESSION  $\frac{\partial \lambda}{\partial z} \propto z$

VLASOV EQUATION ALSO ⇒ ENVELOPE EQUATION

$$\frac{d^2 v_z}{ds^2} = \frac{E_z}{v_z^3} + \frac{3}{2} \frac{g g_0 c}{4\pi \epsilon_0 m v_z^2} \frac{1}{v_z^2} - K(s) v_z$$

KINETIC SOLUTION TO VLASOV EQUATION SATISFYING VMC ENVELOPE EQUATION IS "MUFFET DISTRIBUTION" (ANALOGOUS TO KV).

$$f(z, z') = \frac{3N}{2\pi \epsilon_0} \sqrt{1 - \frac{z^2}{v_z^2} - \frac{v_z^2}{E_z^2} (z' - v_z' z / v_z)^2}$$