

Transverse Centroid and Envelope Descriptions of Beam Evolution*

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USPAS: “Beam Physics with Intense Space-Charge”

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Transverse Centroid and Envelope Model: Outline

Overview

Derivation of Centroid and Envelope Equations of Motion

Centroid Equations of Motion

Envelope Equations of Motion

Matched Envelope Solutions

Envelope Perturbations

Envelope Modes in Continuous Focusing

Envelope Modes in Periodic Focusing

Transport Limit Scaling Based on Envelope Models

Centroid and Envelope Descriptions via 1st order Coupled Moment Equations

Comments:

- ◆ Some of this material related to J.J. Barnard lectures:
 - Transport limit discussions (**Introduction**)
 - Transverse envelope modes (**Continuous Focusing Envelope Modes and Halo**)
 - Longitudinal envelope evolution (**Longitudinal Beam Physics III**)
 - 3D Envelope Modes in a Bunched Beam (**Cont. Focusing Envelope Modes and Halo**)
- ◆ Specific topics will be covered in more detail here

Transverse Centroid and Envelope Model: Detailed Outline

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Overview

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10) Centroid and Envelope Descriptions via 1st Order Coupled Moment

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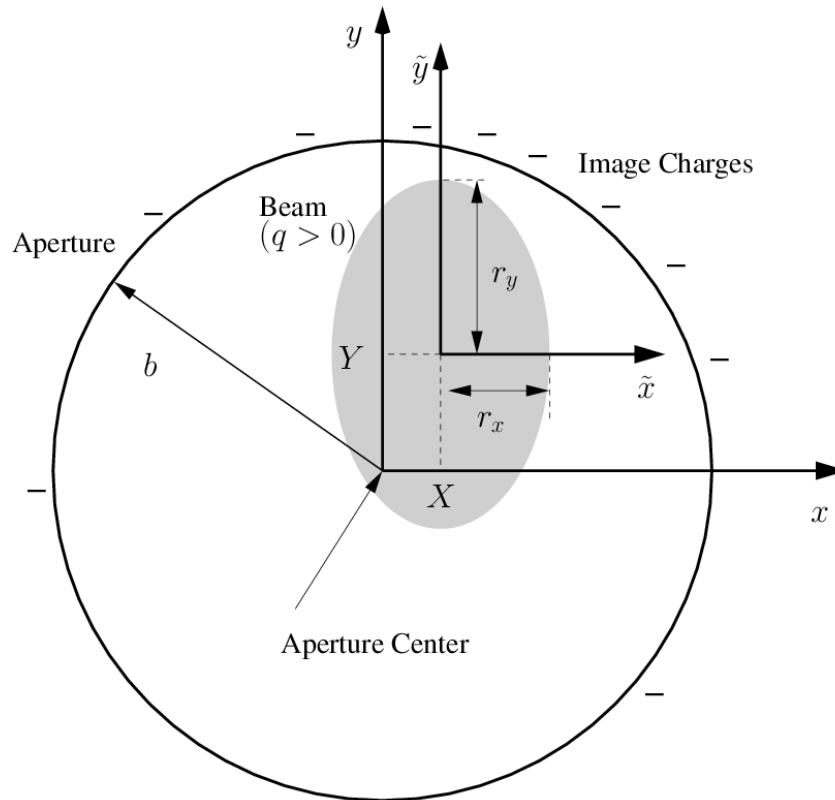
Example Illustration -- Familiar KV Envelope Model

Contact Information

References

S1: Overview

Analyze **transverse centroid and envelope** properties of an unbunched ($\partial/\partial z = 0$) beam



Centroid:

$$X = \langle x \rangle_{\perp}$$

$$Y = \langle y \rangle_{\perp}$$

x- and y-coordinates
of beam center of mass

Envelope: (edge measure)

$$r_x = 2\sqrt{\langle (x - X)^2 \rangle_{\perp}}$$

$$r_y = 2\sqrt{\langle (y - Y)^2 \rangle_{\perp}}$$

x- and y-principal axis radii
of an elliptical beam envelope

◆ Apply to general f but base on uniform density f

Transverse averages:

$$\langle \dots \rangle_{\perp} \equiv \frac{\int d^2 x_{\perp} \int d^2 x'_{\perp} \dots f_{\perp}}{\int d^2 x_{\perp} \int d^2 x'_{\perp} f_{\perp}}$$

Oscillations in the statistical beam centroid and envelope radii are the *lowest-order* collective responses of the beam

Centroid Oscillations: Associated with errors and are purposefully suppressed to the level possible

◆ Error Sources:

- Beam distribution asymmetries
- Dipole bending terms from applied field optics (due to field error or mech misalignment)
- Imperfect mechanical alignment

◆ Exception: When the beam is kicked (insertion or extraction) into or out of a transport channel as is often done in rings

Envelope Oscillations: Can have two components in periodic focusing lattices

1) **Matched Envelope:** Periodic “flutter” synchronized to period of focusing lattice to yield net focusing

◆ Properly tuned flutter essential in Alternating Gradient quadrupole lattices

2) **Mismatched Envelope:** Excursions deviate from matched flutter motion and are seeded/driven by errors

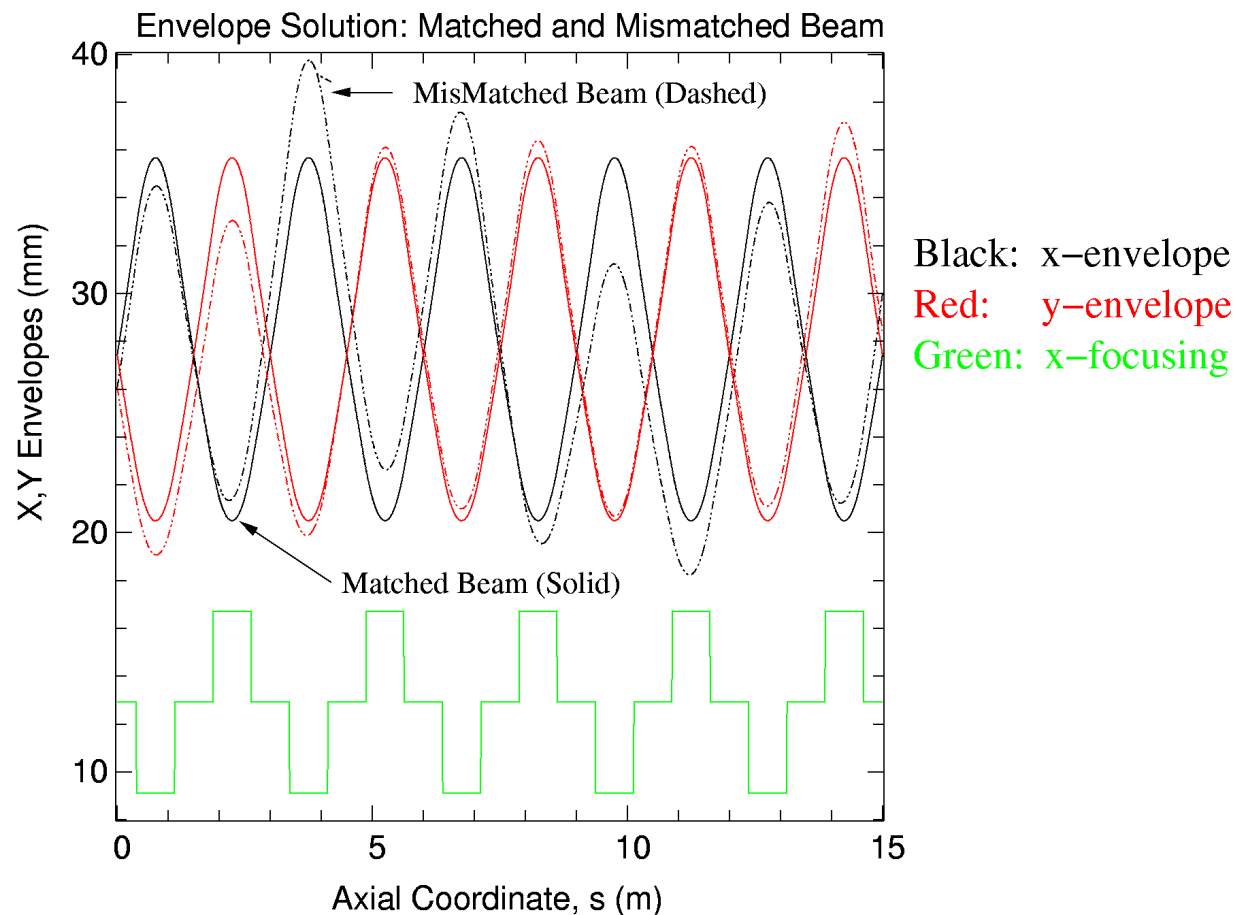
Limiting maximum beam-edge excursions is desired for economical transport

- Reduces cost by Limiting material volume needed to transport an intense beam

Mismatched beams have larger envelope excursions and have more collective stability and beam halo problems since mismatch adds another source of free energy that can drive statistical increases in particle amplitudes

(see: J.J. Barnard lectures on **Envelopes and Halo**)

Example: FODO Quadrupole Transport Channel



◆ Larger machine aperture is needed to confine a mismatched beam

Centroid and Envelope oscillations are the *most important collective modes* of an intense beam

- ◆ Force balances based on matched beam envelope equation **predict scaling of transportable beam parameters**
 - Used to design transport lattices
- ◆ **Instabilities** in beam centroid and/or envelope oscillations **can prevent reliable transport**
 - Parameter locations of instability regions should be understood and avoided in machine design/operation

Although it is *necessary* to avoid envelope and centroid instabilities in designs, it is not alone *sufficient* for effective machine operation

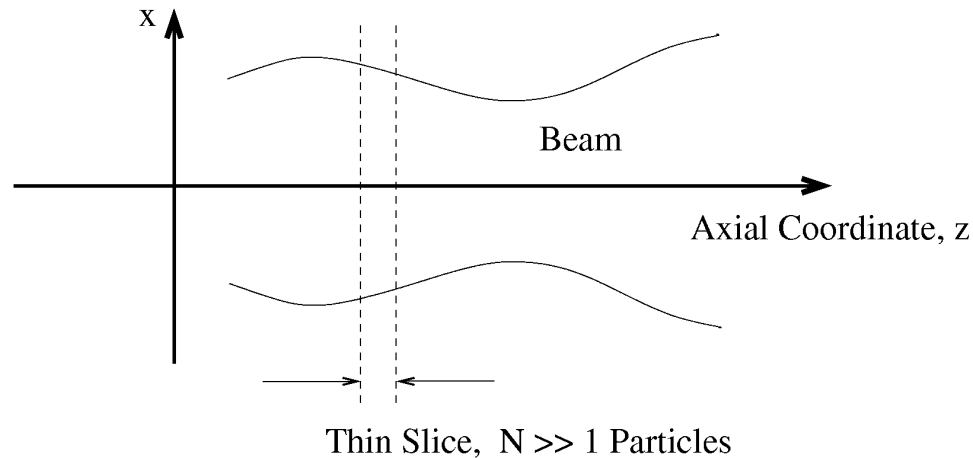
- ◆ Higher-order **kinetic and fluid instabilities** not expressed in the low-order envelope models can **can degrade beam quality and control** and must also be evaluated
 - To be covered (see: S.M. Lund, lectures on **Kinetic Stability**)

S2: Derivation of Transverse Centroid and Envelope Equations of Motion

Analyze centroid and envelope properties of an unbunched ($\partial/\partial z = 0$) beam

Transverse Statistical Averages:

Let N be the number of particles in a thin axial slice of the beam at axial coordinate s .



Averages can be equivalently defined in terms of the discrete **particles** making up the beam or the continuous model transverse Vlasov **distribution function**:

$$\begin{aligned} \text{particles:} \quad \langle \cdots \rangle_{\perp} &\equiv \frac{1}{N} \sum_{i=1}^N \Big|_{\text{slice}} \cdots \\ \text{distribution:} \quad \langle \cdots \rangle_{\perp} &\equiv \frac{\int d^2 x_{\perp} \int d^2 x'_{\perp} \cdots f_{\perp}}{\int d^2 x_{\perp} \int d^2 x'_{\perp} f_{\perp}} \end{aligned}$$

◆ Averages can be generalized to include axial momentum spread

Transverse Particle Equations of Motion

Consistent with earlier analysis [lectures on **Transverse Particle Equations**], take:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + \kappa_x x = - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + \kappa_y y = - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$$\nabla_{\perp}^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = - \frac{\rho}{\epsilon_0}$$

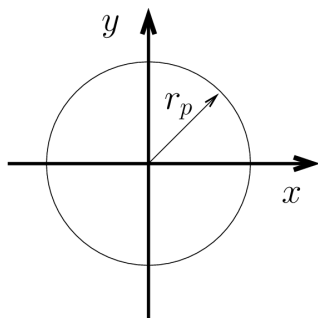
$$\rho = q \int d^2 x'_{\perp} f_{\perp} \quad \phi|_{\text{aperture}} = 0$$

Assume:

- ◆ Unbunched beam
- ◆ No axial momentum spread
- ◆ Linear applied focusing fields described by κ_x, κ_y
- ◆ Possible acceleration, $\gamma_b \beta_b$ need not be constant

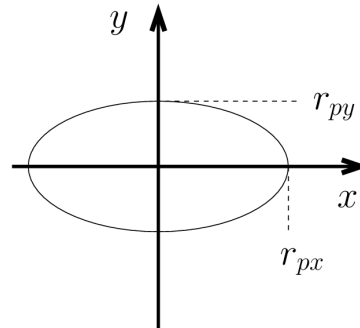
Various apertures are possible influence solution for ϕ . Some simple examples:

Round Pipe



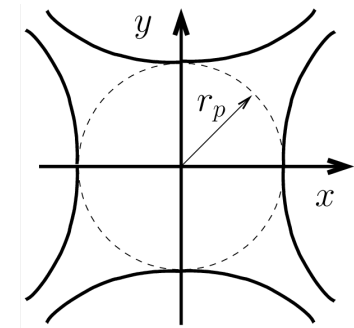
Linac magnetic quadrupoles, acceleration cells,

Elliptical Pipe



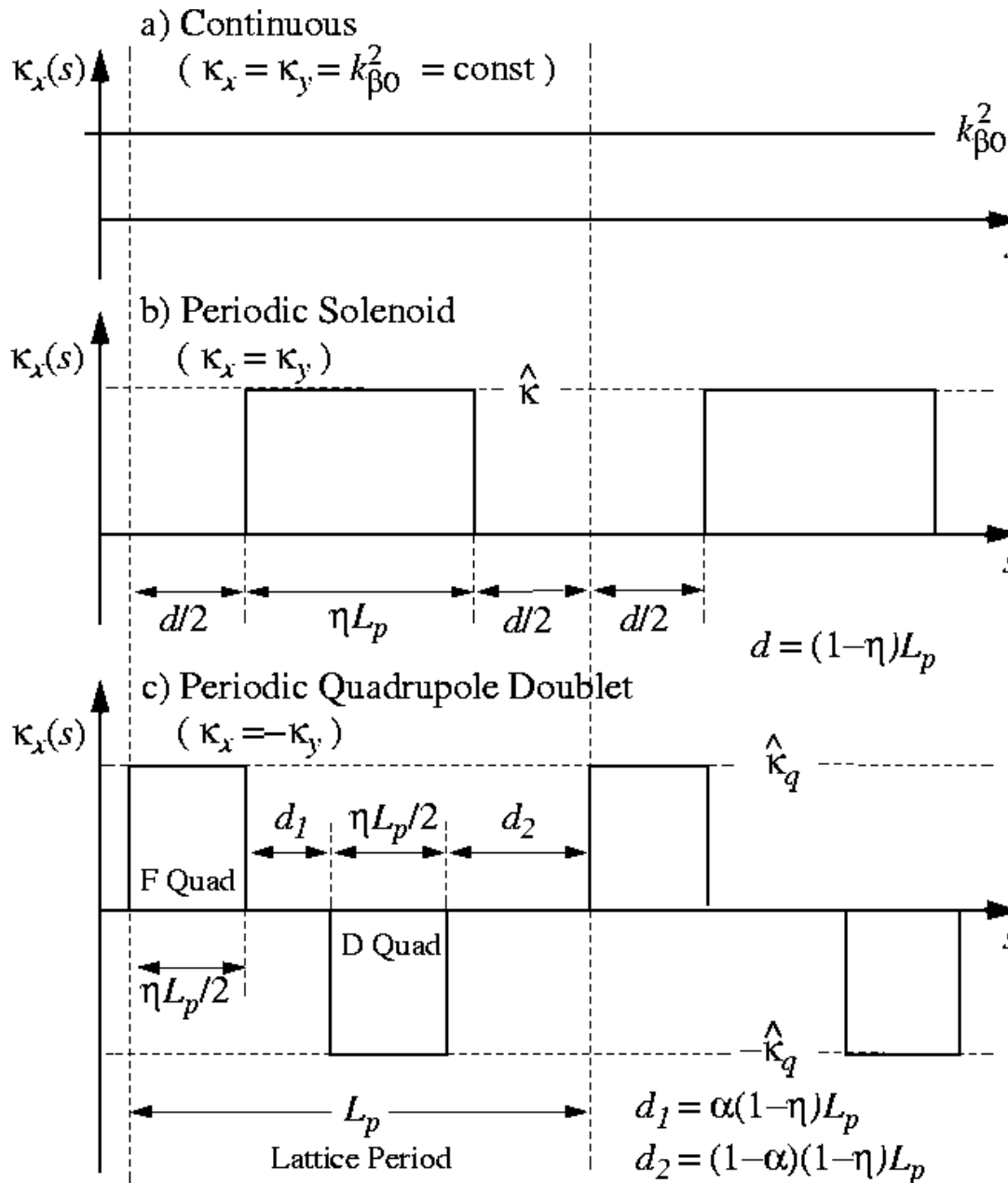
In rings with dispersion: in drifts, magnetic optics,

Hyperbolic Sections



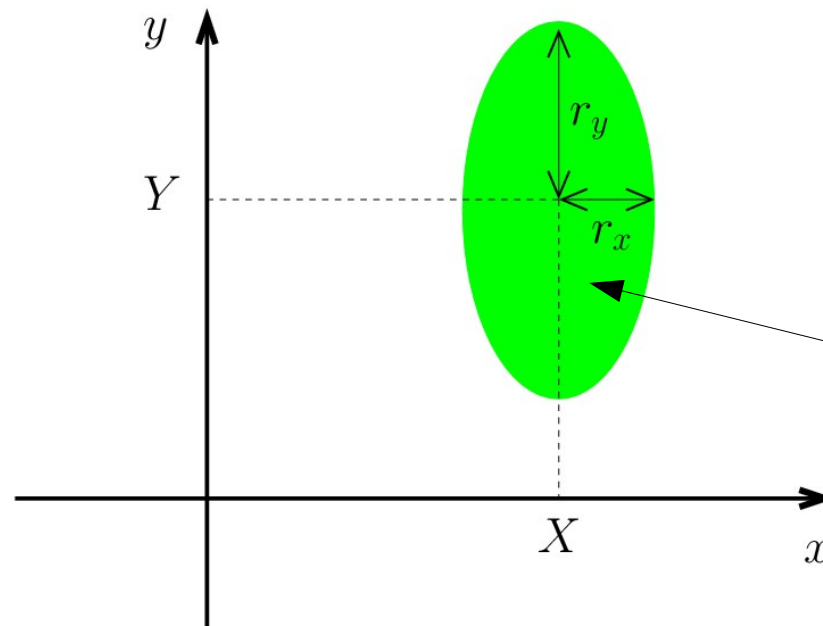
Electric quadrupoles

Review: Focusing lattices we will take in examples: Continuous and piecewise constant periodic solenoid and quadrupole doublet



Distribution Assumptions

To lowest order, linearly focused intense beams are **expected to be nearly uniform in density** within the core of the beam out to an edge where the density falls rapidly to zero



Charge conservation requires:

$$\lambda = \text{const}$$

Uniform density in beam:

$$\rho = \frac{\lambda}{\pi r_x r_y}$$

$$\rho(x, y) = q \int d^2 x'_\perp f_\perp \simeq \begin{cases} \frac{\lambda}{\pi r_x r_y}, & (x - X)^2 / r_x^2 + (y - Y)^2 / r_y^2 < 1 \\ 0, & (x - X)^2 / r_x^2 + (y - Y)^2 / r_y^2 > 1 \end{cases}$$

$$\lambda = q \int d^2 x_\perp \int d^2 x'_\perp f_\perp = \int d^2 x \rho = \text{const}$$

Comments:

- ◆ Nearly uniform density out to a sharp beam edge expected for near equilibrium structure beam with strong space-charge due to Debye screening
 - see: S.M. Lund, lectures on **Transverse Equilibrium Distributions**
- ◆ Simulations support that uniform density model is a good approximation for stable non-equilibrium beams when space-charge is high
- ◆ Assumption of a fixed form of distribution essentially closes the infinite hierarchy of moments that are needed to describe a general beam distribution
 - Need only describe shape/edge and center for uniform density beam to fully specify the distribution!
 - Analogous to closures of fluid theories using assumed equations of state etc.

Self-Field Calculation

Temporarily, we will consider an *arbitrary* beam charge distribution within an arbitrary aperture to formulate the problem.

Electrostatic field of a line charge in free-space

$$\mathbf{E}_{\perp} = \frac{\lambda_0}{2\pi\epsilon_0} \frac{(\mathbf{x}_{\perp} - \tilde{\mathbf{x}})}{|\mathbf{x}_{\perp} - \tilde{\mathbf{x}}|^2}$$

$\lambda_0 =$ line charge

$\mathbf{x}_{\perp} = \tilde{\mathbf{x}} =$ coordinate of charge

Resolve the field of the beam into direct (free space) and image terms:

$$\mathbf{E}_{\perp}^s = -\frac{\partial\phi}{\partial\mathbf{x}_{\perp}} = \mathbf{E}_{\perp}^d + \mathbf{E}_{\perp}^i$$

and superimpose free-space solutions for direct and image contributions

Direct Field

$$\mathbf{E}_{\perp}^d(\mathbf{x}_{\perp}) = \frac{1}{2\pi\epsilon_0} \int d^2\tilde{x}_{\perp} \frac{\rho(\tilde{\mathbf{x}}_{\perp})(\mathbf{x}_{\perp} - \tilde{\mathbf{x}}_{\perp})}{|\mathbf{x}_{\perp} - \tilde{\mathbf{x}}_{\perp}|^2} \quad \rho(\mathbf{x}) = \text{beam charge density}$$

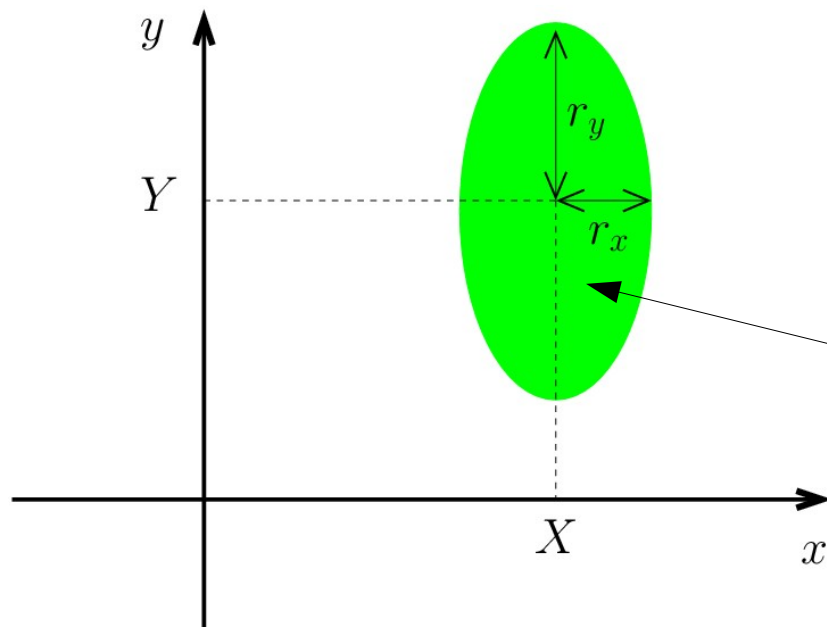
Image Field

$$\mathbf{E}_{\perp}^i(\mathbf{x}_{\perp}) = \frac{1}{2\pi\epsilon_0} \int d^2\tilde{x}_{\perp} \frac{\rho^i(\tilde{\mathbf{x}}_{\perp})(\mathbf{x}_{\perp} - \tilde{\mathbf{x}}_{\perp})}{|\mathbf{x}_{\perp} - \tilde{\mathbf{x}}_{\perp}|^2} \quad \rho^i(\mathbf{x}) = \text{beam image charge density induced on aperture}$$

Direct Field:

The direct field solution for a uniform density beam in free-space was calculated for the KV equilibrium distribution

- see: S.M. Lund, lectures on **Transverse Equilibrium Distributions**



Uniform density in beam:

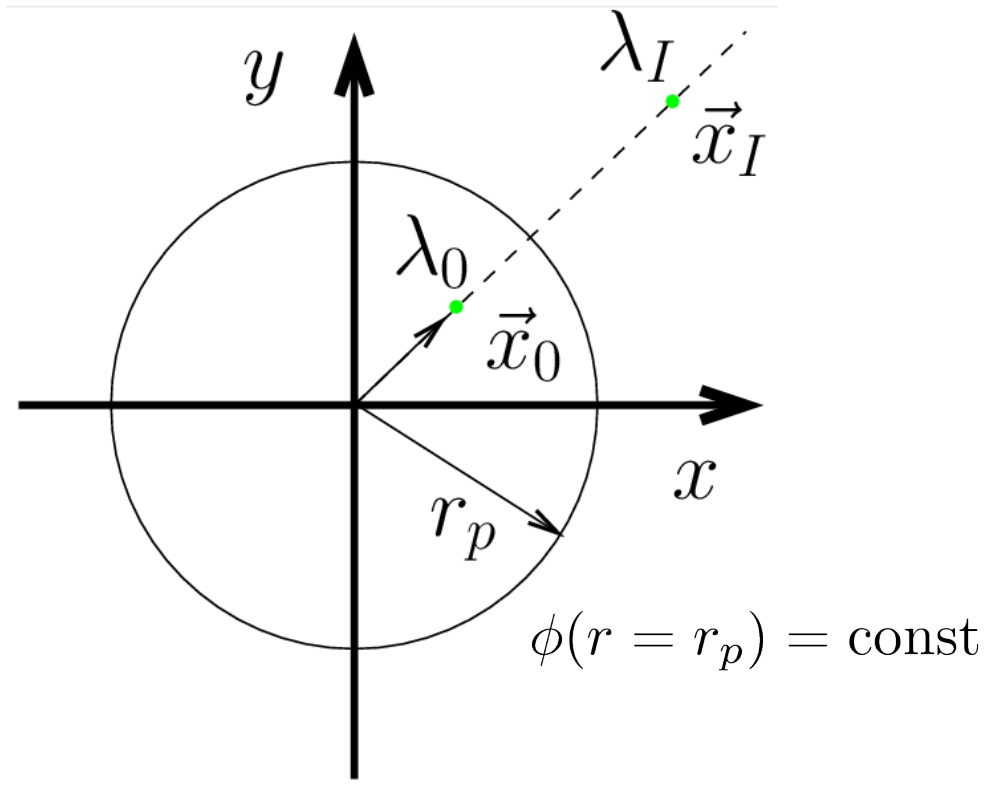
$$\rho = \frac{\lambda}{\pi r_x r_y} = \text{const}$$

$$E_x^d = \frac{\lambda}{\pi \epsilon_0} \frac{x - X}{(r_x + r_y) r_x}$$
$$E_y^d = \frac{\lambda}{\pi \epsilon_0} \frac{y - Y}{(r_x + r_y) r_y}$$

Expressions are valid only within the elliptical density beam -- where they will be applied in taking averages

Image Field:

Image structure depends on the aperture. Assume a round pipe (most common case) for simplicity.



$$\lambda_I = -\lambda_0 \quad \text{image charge}$$

$$\mathbf{x}_I = \frac{r_p^2}{|\mathbf{x}_0|^2} \mathbf{x}_0 \quad \text{image location}$$

Will be derived in the
the problem sets.

superimpose all images of beam:

$$\mathbf{E}_{\perp}^i(\mathbf{x}_{\perp}) = -\frac{1}{2\pi\epsilon_0} \int_{\text{pipe}} d^2\tilde{\mathbf{x}}_{\perp} \frac{\rho(\tilde{\mathbf{x}}_{\perp})(\mathbf{x}_{\perp} - r_p^2\tilde{\mathbf{x}}_{\perp}/|\tilde{\mathbf{x}}_{\perp}|^2)}{|\mathbf{x}_{\perp} - r_p^2\tilde{\mathbf{x}}_{\perp}/|\tilde{\mathbf{x}}_{\perp}|^2|^2}$$

◆ Difficult to calculate even for ρ corresponding to a uniform density beam

Examine limits of the image field to build intuition on the range of properties:

1) Line charge along x -axis:

choose coordinates to
make true

$$\rho(\mathbf{x}_\perp) = \lambda \delta(\mathbf{x}_\perp - X \hat{\mathbf{x}})$$

Plug this density in the image charge expression for a round-pipe aperture:

- ◆ Need only evaluate at $\mathbf{x}_\perp = X \hat{\mathbf{x}}$ since beam is at that location

$$\mathbf{E}_\perp^i(\mathbf{x}_\perp = X \hat{\mathbf{x}}) = \frac{\lambda}{2\pi\epsilon_0(r_p^2/X - X)} \hat{\mathbf{x}}$$

- ◆ Generates **nonlinear field** at position of direct charge
- ◆ Field creates **attractive force** between direct and image charge

2) Centered, uniform density elliptical beam:

Expand using complex coordinates starting from the general image expression:

$$\underline{E}^i = E_y^i + iE_x^i = \sum_{n=2,4,\dots}^{\infty} \underline{c}_n \underline{z}^{n-1} \quad \underline{c}_n = \frac{i}{2\pi\epsilon_0} \int_{\text{pipe}} d^2x_{\perp} \rho(\mathbf{x}_{\perp}) \frac{(x - iy)^n}{r_p^{2n}}$$

$$\underline{z} = x + iy \quad = \frac{i\lambda n!}{2\pi\epsilon_0 2^n (n/2 + 1)! (n/2)!} \left(\frac{r_x^2 - r_y^2}{r_p^4} \right)^{n/2}$$

$$i = \sqrt{-1}$$

The linear ($n = 2$) components of this expansion give:

$$E_x^i = \frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} x, \quad E_y^i = -\frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} y$$

- ◆ Rapidly vanish (higher order terms more rapid) as beam becomes more round
- ◆ Case will be analyzed further in the problem sets

3) Uniform density elliptical beam with a small displacement along the x -axis:

$$Y = 0 \quad |X|/r_p \ll 1$$

Expand using complex coordinates starting from the general image expression:

- ◆ Use complex coordinates to simplify calculation

E.P. Lee, E. Close, and L. Smith, Nuclear Instruments and Methods, 1126 (1987)

- ◆ Expressions become even more complicated with simultaneous x - and y -displacements and more complicated aperture geometries

Leading
Order
Image
Fields

$$E_x^i = \frac{\lambda}{2\pi\epsilon_0 r_p^2} [f(x - X) + gX] + \Theta \left(\frac{X}{r_p} \right)^3$$

$$E_y^i = -\frac{\lambda}{2\pi\epsilon_0 r_p^2} f y + \Theta \left(\frac{X}{r_p} \right)^3$$

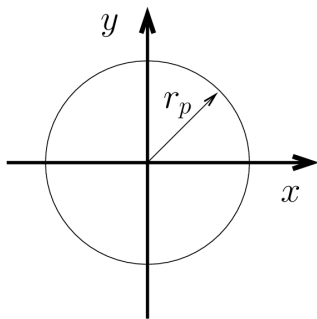
$$f = \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{X^2}{r_p^2} \left[1 + \frac{3}{2} \left(\frac{r_x^2 - r_y^2}{r_p^2} \right) + \frac{3}{8} \left(\frac{r_x^2 - r_y^2}{r_p^2} \right)^2 \right]$$

$$g = 1 + \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{X^2}{r_p^2} \left[1 + \frac{3}{4} \left(\frac{r_x^2 - r_y^2}{r_p^2} \right) + \frac{1}{8} \left(\frac{r_x^2 - r_y^2}{r_p^2} \right)^2 \right]$$

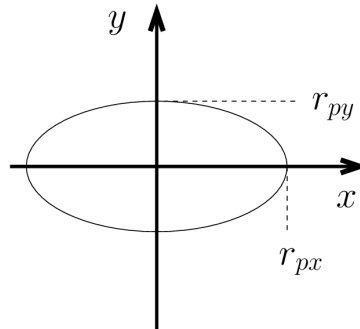
Comments on images:

- ◆ Sign is generally such that it will **tend to increase beam displacements**
 - Also (usually) weak linear focusing corrections for an elliptical beam
- ◆ Can be very **difficult to calculate explicitly**
 - Even for simple case of circular pipe
 - Special cases of simple geometry formulas can give idea on scaling
 - Generally suppress just by making the beam small relative to characteristic aperture dimensions and keeping the beam steered near-axis
 - Simulations typically applied
- ◆ **Depend strongly on the aperture geometry**
 - Generally varies as a function of s in the machine aperture changes and/or beam symmetries evolve

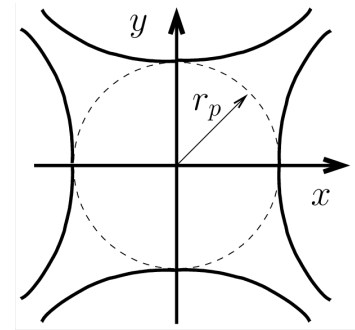
Round Pipe



Elliptical Pipe



Hyperbolic Sections



Coupled centroid and envelope equations of motion

Consistent with the assumed structure of the distribution (uniform density elliptical beam), denote:

Beam Centroid:

$$X \equiv \langle x \rangle_{\perp} \quad X' = \langle x' \rangle_{\perp}$$

$$Y \equiv \langle y \rangle_{\perp} \quad Y' = \langle y' \rangle_{\perp}$$

Coordinates with respect to centroid:

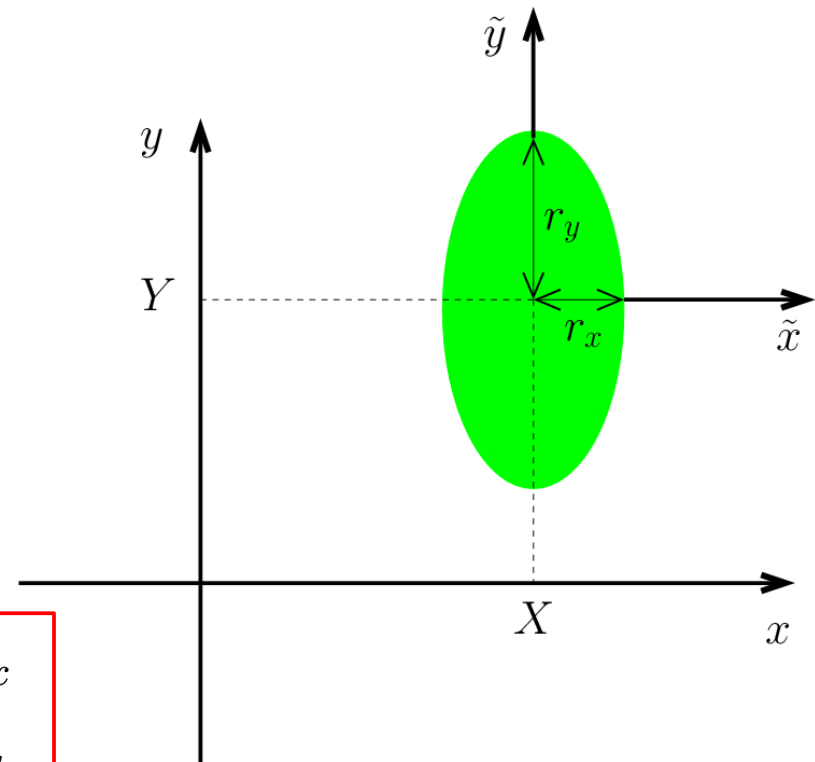
$$\tilde{x} \equiv x - X \quad \tilde{x}' = x' - X'$$

$$\tilde{y} \equiv y - Y \quad \tilde{y}' = y' - Y'$$

Envelope Edge Radii:

$$r_x \equiv 2\sqrt{\langle \tilde{x}^2 \rangle_{\perp}} \quad r'_x = \langle \tilde{x}\tilde{x}' \rangle_{\perp} / r_x$$

$$r_y \equiv 2\sqrt{\langle \tilde{y}^2 \rangle_{\perp}} \quad r'_y = \langle \tilde{y}\tilde{y}' \rangle_{\perp} / r_y$$



With the *assumed* uniform elliptical beam, **all moments** can be calculated in terms of: X, Y, r_x, r_y

- Such truncations follow whenever the form of the distribution is “frozen”

Derive centroid equations: First use the self-field resolution for a uniform density beam, then the equations of motion for a particle within the beam are:

$$\begin{aligned}
 x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + \kappa_x x - \frac{2Q}{(r_x + r_y)r_x} (x - X) &= \frac{q}{m\gamma_b^3 \beta_b^2 c^2} E_x^i \\
 y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + \kappa_y y - \frac{2Q}{(r_x + r_y)r_y} (y - Y) &= \frac{q}{m\gamma_b^3 \beta_b^2 c^2} E_y^i
 \end{aligned}$$

Direct Terms
Image Terms

Perveance:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3 \beta_b^2 c^2} \quad (\text{not necessarily constant if beam accelerates})$$

average equations using: $\langle x' \rangle_{\perp} = \langle x \rangle'_{\perp} = X'$ etc., to obtain:

Centroid Equations:

$$\begin{aligned}
 X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \kappa_x X &= Q \left[\frac{2\pi\epsilon_0}{\lambda} \langle E_x^i \rangle_{\perp} \right] \\
 Y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} Y' + \kappa_y Y &= Q \left[\frac{2\pi\epsilon_0}{\lambda} \langle E_y^i \rangle_{\perp} \right]
 \end{aligned}$$

Note: the electric image field will cancel the coefficient $2\pi\epsilon_0/\lambda$

♦ $\langle E_x^i \rangle_{\perp}$ will generally depend on: X , Y and r_x , r_y

To derive equations of motion for the envelope radii, first subtract the centroid equations from the particle equations of motion ($\tilde{x} \equiv x - X$) to obtain:

$$\tilde{x}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{x}' + \kappa_x \tilde{x} - \frac{2Q\tilde{x}}{(r_x + r_y)r_x} = \frac{q}{m\gamma_b^2 \beta_b^2 c^2} [E_x^i - \langle E_x^i \rangle_{\perp}]$$

$$\tilde{y}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{y}' + \kappa_y \tilde{y} - \frac{2Q\tilde{y}}{(r_x + r_y)r_x} = \frac{q}{m\gamma_b^2 \beta_b^2 c^2} [E_y^i - \langle E_y^i \rangle_{\perp}]$$

Differentiate the equation for the envelope radius (y-equations analogous):

$$r_x = 2\langle \tilde{x}^2 \rangle_{\perp}^{1/2} \longrightarrow r'_x = \frac{2\langle \tilde{x}\tilde{x}' \rangle_{\perp}}{\langle \tilde{x}^2 \rangle_{\perp}} = \frac{4\langle \tilde{x}\tilde{x}' \rangle_{\perp}}{r_x}$$

Define (motivated the KV equilibrium results) a statistical rms edge emittance:

$$\varepsilon_x \equiv 4\varepsilon_{x,\text{rms}} \equiv 4 \left[\langle \tilde{x}^2 \rangle_{\perp} \langle \tilde{x}'^2 \rangle_{\perp} - \langle \tilde{x}\tilde{x}' \rangle_{\perp}^2 \right]^{1/2}$$

Differentiate the equation for r'_x again and use the emittance definition:

$$r''_x = 4 \frac{\langle \tilde{x}\tilde{x}'' \rangle_{\perp}}{r_x} + \frac{16[\langle \tilde{x}^2 \rangle_{\perp} \langle \tilde{x}'^2 \rangle_{\perp} - \langle \tilde{x}\tilde{x}' \rangle_{\perp}^2]}{r_x^3}$$

$$= 4 \frac{\langle \tilde{x}\tilde{x}'' \rangle_{\perp}}{r_x} + \frac{\varepsilon_x^2}{r_x^3}$$

and then employ the equations of motion to eliminate \tilde{x}'' in $\langle \tilde{x}\tilde{x}'' \rangle_{\perp}$ to obtain:

Envelope Equations:

$$r_x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 8Q \left[\frac{\pi \epsilon_0}{\lambda} \langle \tilde{x} E_x^i \rangle_{\perp} \right]$$
$$r_y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r_y' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 8Q \left[\frac{\pi \epsilon_0}{\lambda} \langle \tilde{y} E_y^i \rangle_{\perp} \right]$$

- ◆ $\langle \tilde{x} E_x^i \rangle_{\perp}$ will generally depend on: X , Y and r_x , r_y

Comments on Centroid/Envelope equations:

- ◆ Centroid and envelope equations are *coupled* and must be solved simultaneously when image terms on the RHS cannot be neglected
- ◆ Image terms contain nonlinear terms that can be difficult to evaluate explicitly
 - Aperture geometry changes image correction
- ◆ The formulation is not self-consistent because a frozen form (uniform density) charge profile is assumed
 - Uniform density choice motivated by KV results and Debye screening
see: S.M. Lund, lectures on **Transverse Equilibrium Distributions**
 - The assumed distribution form not evolving represents a fluid model closure
 - Generally find with simulations that uniform density frozen form distribution models can provide reasonably accurate approximate models for

Comments on Centroid/Envelope equations (Continued):

- ♦ Constant (normalized when accelerating) emittances are generally assumed
 - See: S.M. Lund, lectures on **Transverse Particle Equations**

$\beta_b, \gamma_b, \lambda$ s-variation set by acceleration schedule

$$\begin{aligned}\varepsilon_{nx} &= \gamma_b \beta_b \varepsilon_x = \text{const} \\ \varepsilon_{ny} &= \gamma_b \beta_b \varepsilon_y = \text{const}\end{aligned} \quad \longrightarrow \quad \text{used to calculate } \varepsilon_x, \varepsilon_y$$

$$Q = \frac{q\lambda}{2\pi m \epsilon_0 \gamma_b^3 \beta_b^2 c^2}$$

S3: Centroid Equations of Motion

Single Particle Limit: Oscillation and Stability Properties

Neglect image charge terms, then the centroid equation of motion becomes:

$$X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \kappa_x X = 0$$
$$Y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} Y' + \kappa_y Y = 0$$

- ◆ Usual **Hill's equation** with generalized acceleration term
- ◆ Single particle form. Apply results from S.M. Lund lectures on **Transverse Particle Equations**: phase amplitude methods, Courant-Snyder invariants, and stability bounds, ...

Assume that applied lattice focusing is tuned for constant phase advances and/or that acceleration is weak and can be neglected. Then single particle stability results give immediately:

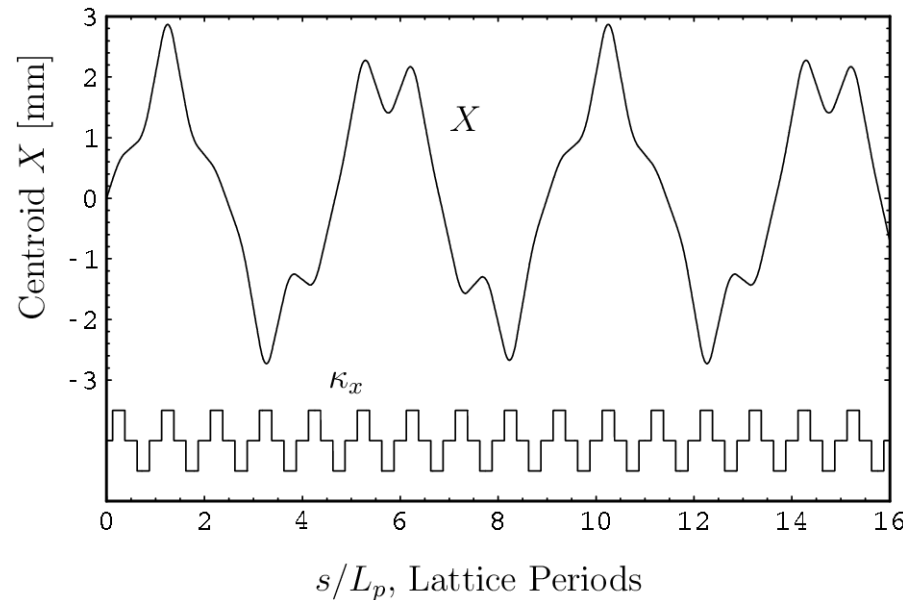
$$\sigma_{0x} < 180^\circ$$
$$\sigma_{0y} < 180^\circ$$

centroid stability, 1st stability condition

/// Example: FODO channel centroid evolution

Mid-drift
launch:

$$X(0) = 1 \text{ mm}$$
$$X'(0) = 1 \text{ mrad}$$



lattice/beam
parameters:

$$\beta_b = \text{const}$$

$$\sigma_{0x} = 80^\circ$$

$$L_p = 0.5 \text{ m}$$

$$\eta = 0.5$$

- ◆ Centroid exhibits expected characteristic stable betatron oscillations

///

Effect of Driving Errors

The reference orbit is **ideally tuned for zero centroid excursions**. But there will *always* be driving errors that can cause the centroid oscillations to accumulate with beam propagation distance:

$$X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \frac{G_n}{G_0} \kappa_q(s) X = \frac{G_n}{G_0} \kappa_q(s) \Delta_{xn}$$

$\frac{G_n}{G_0} =$ *n*th quadrupole gradient error (unity for no error; s-varying)

$\Delta_{xn} =$ *n*th quadrupole transverse displacement error (s-varying)

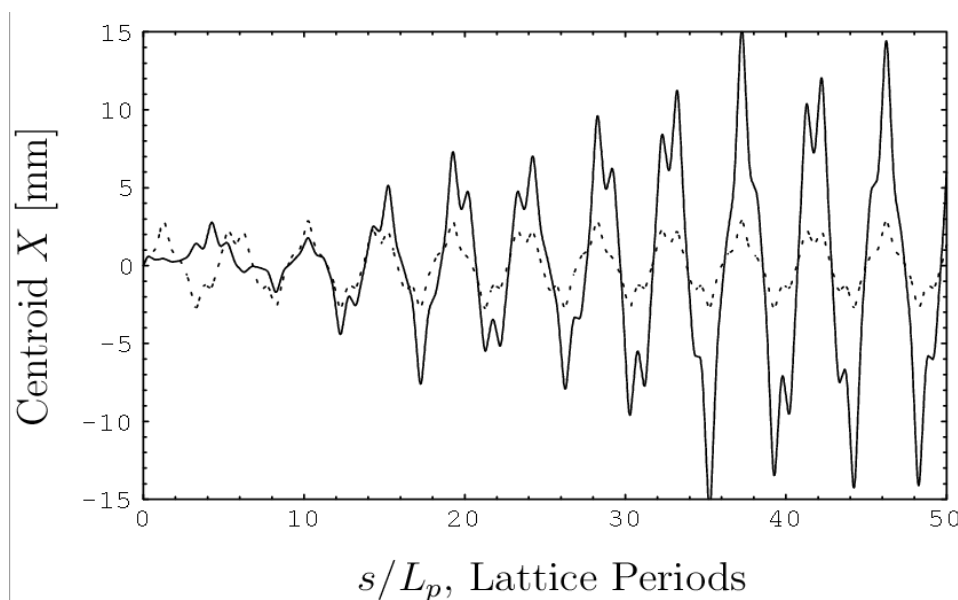
/// Example: FODO channel centroid with quadrupole displacement errors

$$\frac{G_n}{G_0} = 1$$

$$\Delta_{xn} = [-0.5, 0.5] \text{ mm}$$

(uniform dist)

same lattice
as previous



solid – with errors
dashed – no errors

///

Errors will result in a **characteristic random walk** increase in oscillation amplitude due to the (generally random) driving terms.

Control by:

- ◆ Synthesize small applied dipole fields to regularly steer the centroid back on-axis to the reference trajectory: $X = 0 = Y$, $X' = 0 = Y'$
- ◆ Fabricate and align focusing elements with higher precision
- ◆ Employ a sufficiently large aperture to contain the oscillations and limit detrimental nonlinear image charge effects

Economics dictates the optimal strategy

- Usually sufficient control achieved by a combination of methods

Effects of Image Charges

Model the beam as a displaced line-charge in a circular aperture. Then using the previously derived image charge field, the equations of motion reduce to:

$$X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \kappa_x X = \frac{QX}{r_p^2 - X^2}$$

examine oscillation
along x-axis

$$\frac{QX}{r_p^2 - X^2} \simeq \frac{Q}{r_p^2} X + \frac{Q}{r_p^4} X^3$$

linear correction

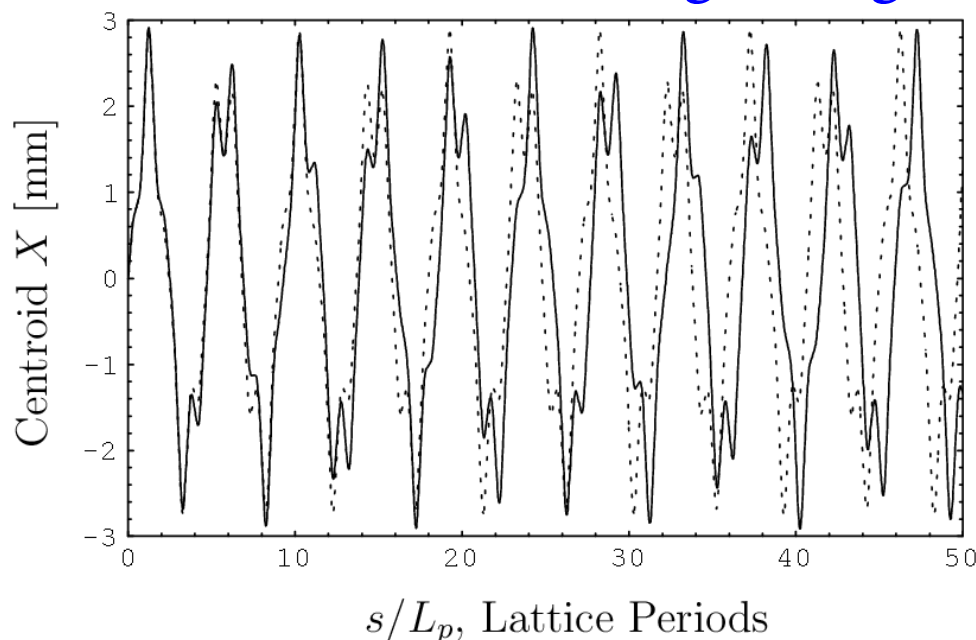
Nonlinear correction (smaller)

Example: FODO channel centroid with image charge corrections

$$r_p = 30 \text{ mm}$$

$$Q = 2 \times 10^{-4}$$

same lattice
as previous



solid - with images
dashed - no images

Main effect of images is generally an **accumulated phase error of the centroid orbit** since, generally the centroid error oscillations are not “matched” orbits and errors are not regularly “undone”

- ◆ This will complicate extrapolations of errors over many lattice periods

Control by:

- ◆ Keeping centroid displacements X, Y small by correcting
- ◆ Make aperture (pipe radius) larger

General Comments:

- ◆ Images contributions to centroid excursions generally less problematic than misalignment errors in focusing elements
- ◆ More detailed analysis show that the coupling of the envelope radii r_x, r_y to the centroid evolution in X, Y is often weak
- ◆ Fringe fields are more important for accurate calculation of centroid orbits since orbits are not part of a matched lattice
 - Non-ideal orbits are poorly tuned to lattice and become more sensitive to the precise phase of impulses
- ◆ Over long path lengths many nonlinear terms can influence results
- ◆ Lattice errors are not often known so one must often analyze characteristic error distributions to see if centroids measured are consistent with expectations

S4: Envelope Equations of Motion

Overview: Reduce equations of motion for r_x, r_y

- ◆ Generally found that couplings to centroid coordinates X, Y are weak
 - Centroid ideally zero in a well tuned system
- ◆ Envelope eqns are most important in designing transverse focusing systems
 - Expresses average radial force balance (see following discussion)
 - Can be difficult to analyze analytically for scaling properties
 - “Systems” codes generally written using envelope equations, stability criteria, and practical engineering constraints
- ◆ Instabilities of the envelope equations in periodic focusing lattices must be avoided in machine operation
 - Instabilities are strong and real: not washed out with realistic distributions without frozen form
 - Represent lowest order “KV” modes of a full kinetic theory
- ◆ Previous derivation of envelope equations relied on Courant-Snyder invariants in linear applied and self-fields. Analysis shows that the same force balances result for a uniform elliptical beam with no image couplings.
 - Debye screening arguments suggest assumed uniform density model taken should be a good approximation for intense space-charge

KV/rms Envelope Equations: Properties of Terms

The envelope equation reflects low-order force balances:

$$\begin{array}{cccccc}
 r''_x & + & \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r'_x & + & \kappa_x r_x & - & \frac{2Q}{r_x + r_y} & - & \frac{\varepsilon_x^2}{r_x^3} & = & 0 \\
 r''_y & + & \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r'_y & + & \kappa_y r_y & - & \frac{2Q}{r_x + r_y} & - & \frac{\varepsilon_y^2}{r_y^3} & = & 0
 \end{array}$$

Terms: **Lattice** **Lattice** **Perveance** **Emittance**

Applied Acceleration Applied Focusing Space-Charge Defocusing Thermal Defocusing

The “acceleration schedule” specifies both $\gamma_b\beta_b$ and λ then the equations are integrated with:

$$\begin{array}{l}
 \gamma_b\beta_b\varepsilon_x = \text{const} \\
 \gamma_b\beta_b\varepsilon_y = \text{const}
 \end{array}$$

normalized emittance conservation

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2}$$

specified perveance

Reminder: It was shown for a coasting beam that the envelope equations remain valid for elliptic charge densities suggesting more general validity [Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971), J.J. Barnard, **Intro. Lectures**]

For any beam with **elliptic symmetry** charge density in each transverse slice:

$$\rho = \rho \left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \right)$$

Based on:

$$\left\langle x \frac{\partial \phi}{\partial x} \right\rangle_{\perp} = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

the KV envelope equations

see J.J. Barnard, **Intro. Lectures**

$$r_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\epsilon_x^2(s)}{r_x^3(s)} = 0$$

$$r_y''(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\epsilon_y^2(s)}{r_y^3(s)} = 0$$

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \quad \lambda = q \int d^2x_{\perp} \rho = \text{const}$$

$$r_x = 2 \langle x^2 \rangle_{\perp}^{1/2}$$

$$\epsilon_x = 4 [\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2]^{1/2}$$

$$r_y = 2 \langle y^2 \rangle_{\perp}^{1/2}$$

$$\epsilon_y = 4 [\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}^2]^{1/2}$$

♦ Evolution changes often small in ϵ_x, ϵ_y

Properties of Envelope Equation Terms:

Applied Focusing: $\kappa_x r_x, \kappa_y r_y$ **and Acceleration:** $\frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_x, \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_y$

- ◆ Analogous to single particle orbit terms
- ◆ Contributions to beam envelope essentially the same as in single particle case
- ◆ Have strong s dependence, *can be both focusing and defocusing*
 - Act only in focusing elements and acceleration gaps

Perveance: $\frac{2Q}{r_x + r_y}$

- ◆ Acts continuously in s , *always defocusing*
- ◆ Becomes stronger (relatively to other terms) when the beam expands in cross-sectional area

Emittance: $\frac{\epsilon_x^2}{r_x^3}$

- ◆ Acts continuously in s , *always defocusing*
- ◆ Becomes stronger (relatively to other terms) when the beam becomes small in cross-sectional area
- ◆ Scaling makes clear why it is necessary to inhibit emittance growth for applications where small spots are desired on target

As the beam expands, the perveance term will eventually dominate:

[see: S.M. Lund and B. Bukh, PRSTAB 7, 024801 (2004)]

Free expansion ($\kappa_x = \kappa_y = 0$)

Initial conditions:

$$r_x(s_i) = r_y(s_i)$$

$$r'_x(s_i) = r'_y(s_i) = 0$$

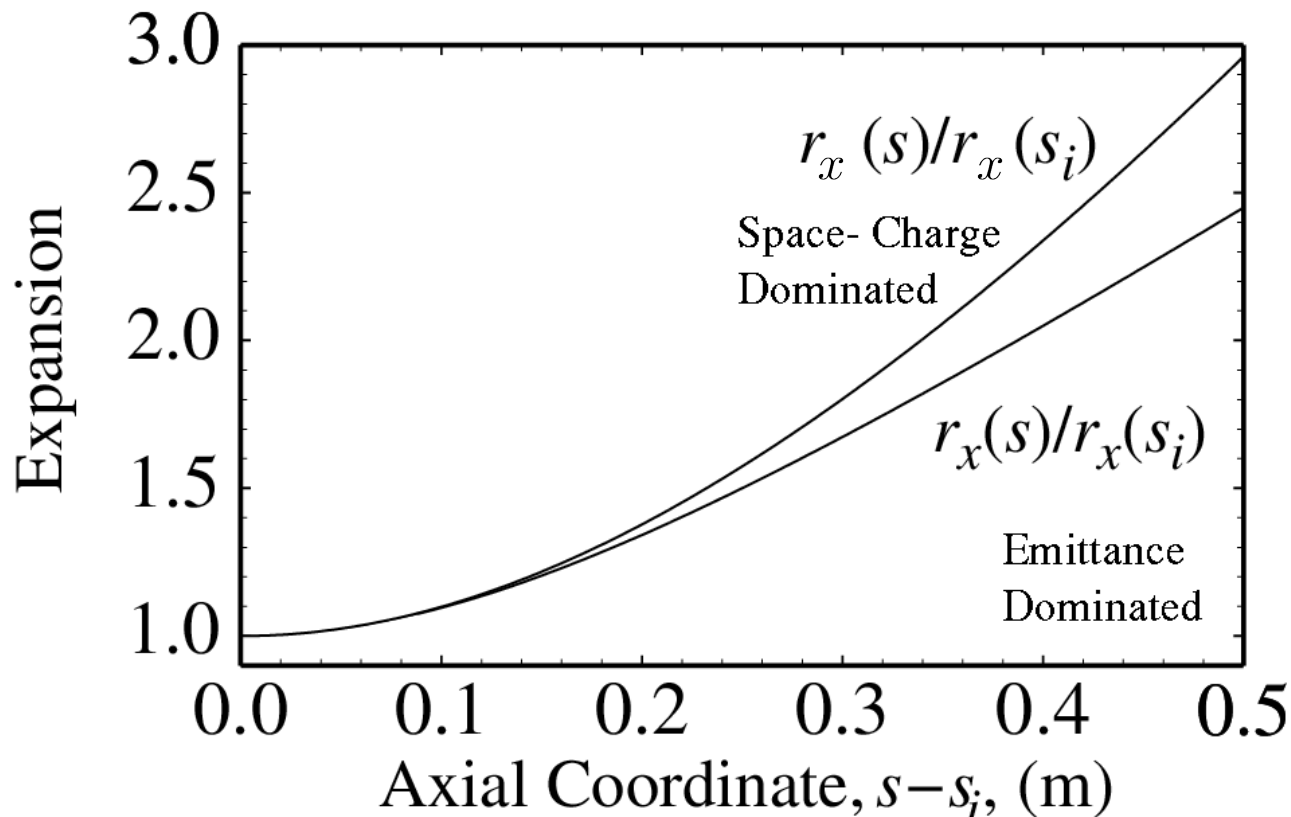
$$\frac{Q}{r_x(s_i)} = \frac{\epsilon_x^2}{r_x^3(s_i)}$$

$$Q = \frac{\epsilon_x^2}{r_x^2(s_i)} = 10^{-3}$$

Cases:

Space-Charge Dominated: $\epsilon_x = 0$

Emittance Dominated: $Q = 0$



See PRSTAB article:
solution is analytical in
bounding limits shown

Parameters are chosen such
that initial defocusing
forces in two limits are
equal

S5: Matched Envelope Solution:

Neglect acceleration ($\gamma_b \beta_b = \text{const}$) or use transformed variables:

$$r_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$

$$r_y''(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$

$$r_x(s + L_p) = r_x(s) \quad r_x(s) > 0$$

$$r_y(s + L_p) = r_y(s) \quad r_y(s) > 0$$

Matching involves finding specific initial conditions for the envelope to have the **periodicity of the lattice**:

Find Values of:

$$\begin{matrix} r_x(s_i) & r_x'(s_i) \\ r_y(s_i) & r_y'(s_i) \end{matrix}$$



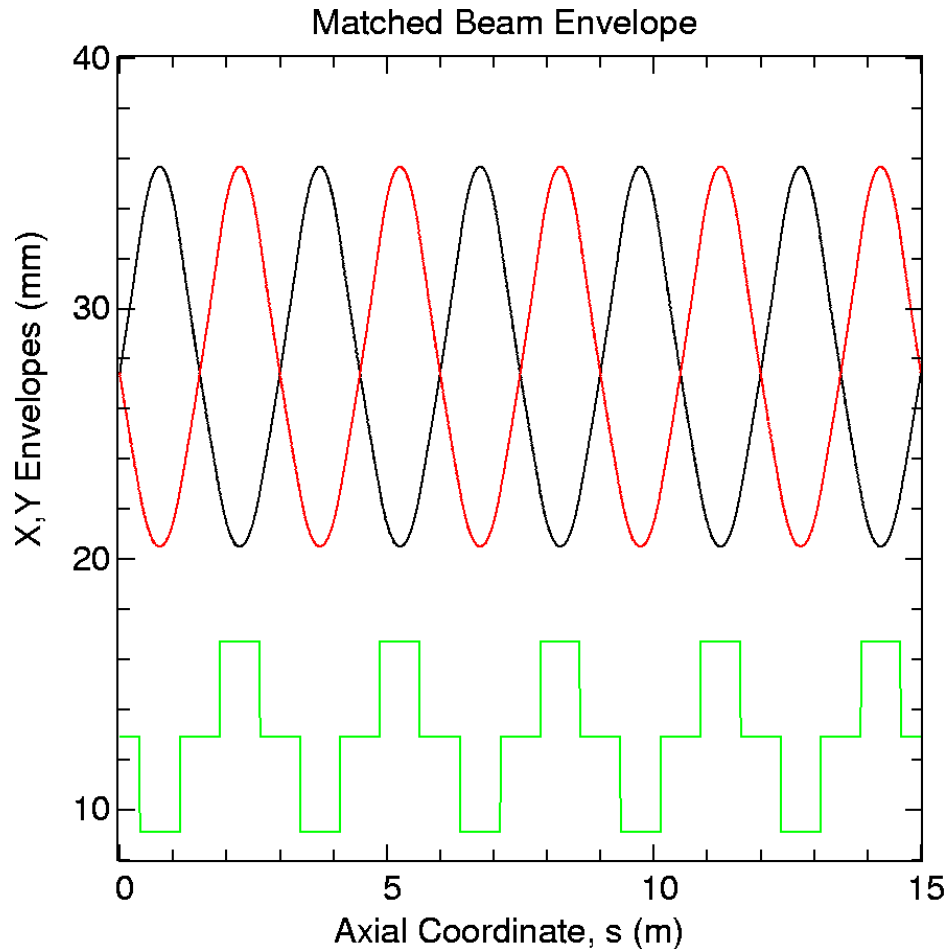
Such That:

$$\begin{matrix} r_x(s_i + L_p) = r_x(s_i) & r_x'(s_i + L_p) = r_x'(s_i) \\ r_y(s_i + L_p) = r_y(s_i) & r_y'(s_i + L_p) = r_y'(s_i) \end{matrix}$$

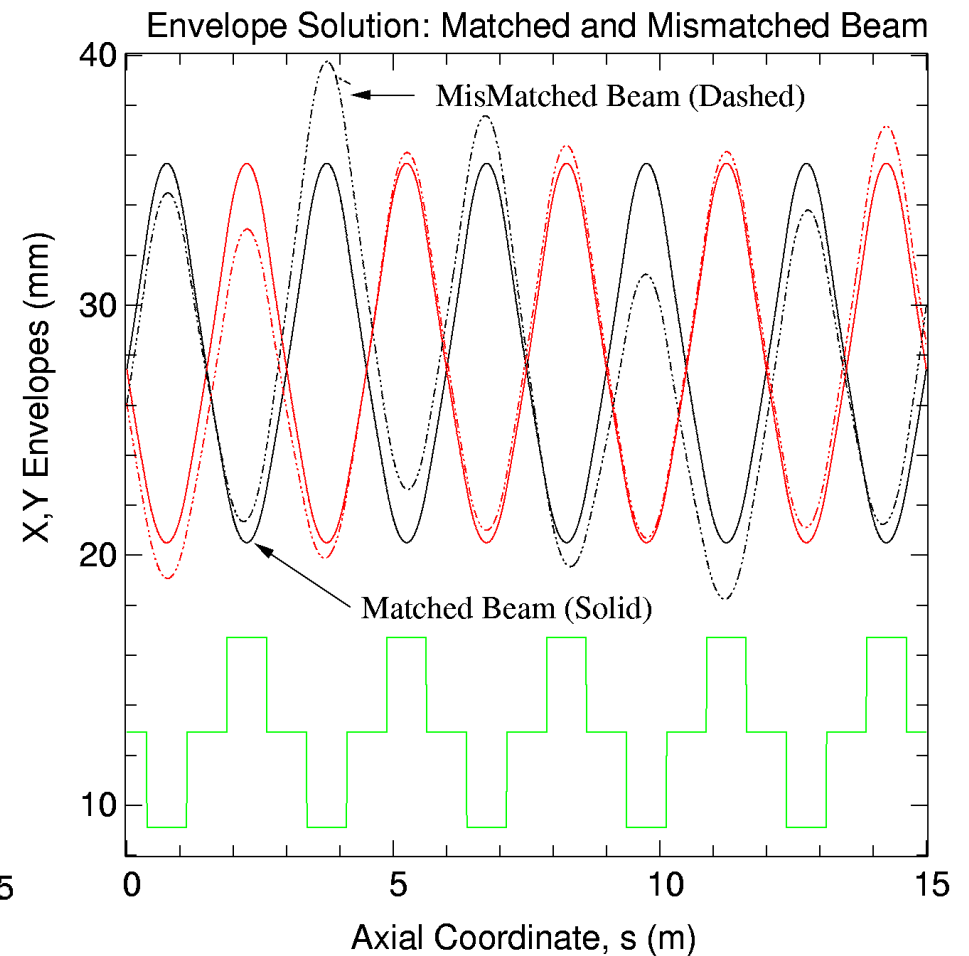
- ◆ Typically constructed with numerical root finding from estimated/guessed values
 - Can be surprisingly difficult for complicated lattices and/or strong space-charge
- ◆ Iterative technique developed to numerically calculate without root finding
[S.M. Lund, S. Chilton and E.P. Lee, PRSTAB **9**, 064201 (2006)]

Typical Matched vs Mismatched solution for FODO channel:

Matched



Mismatched



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

- ◆ Matching tends to exploit optics most efficiently to maintain confinement

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

$$r_x(s + L_p) = r_x(s)$$

$$r_y(s + L_p) = r_y(s)$$

$$\varepsilon_x = \varepsilon_y$$

Parameters

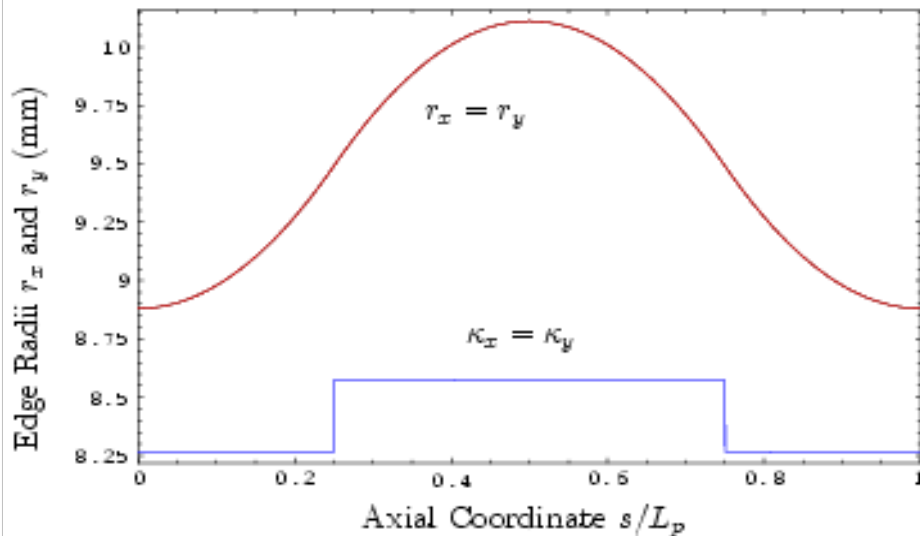
$$L_p = 0.5 \text{ m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$

$$\varepsilon_x = 50 \text{ mm-mrad}$$

$$\sigma/\sigma_0 = 0.2$$

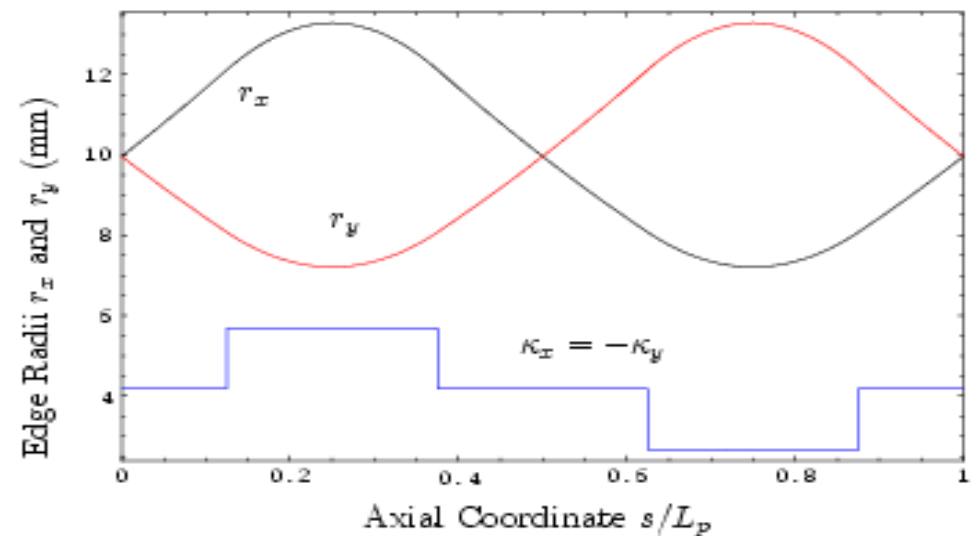
Solenoidal Focusing

$$(Q = 6.6986 \times 10^{-4})$$



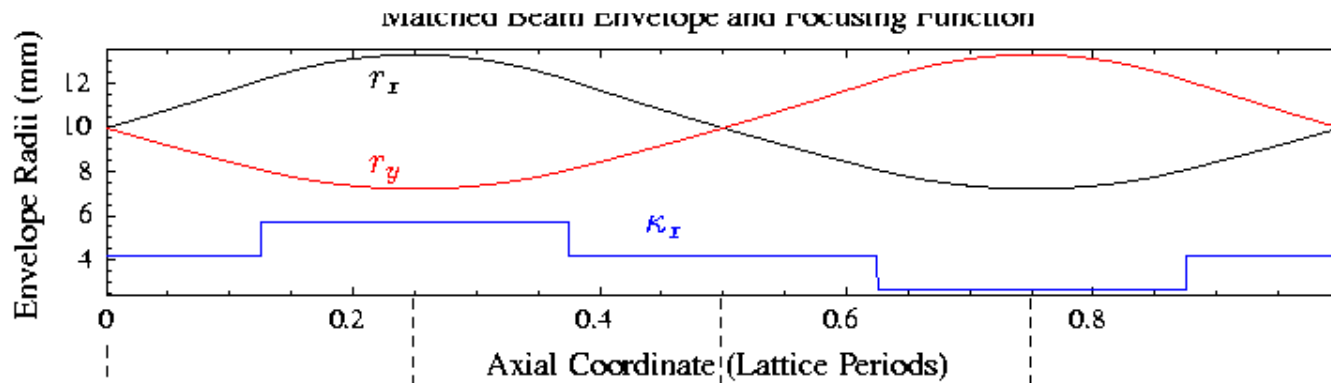
FODO Quadrupole Focusing

$$(Q = 6.5614 \times 10^{-4})$$

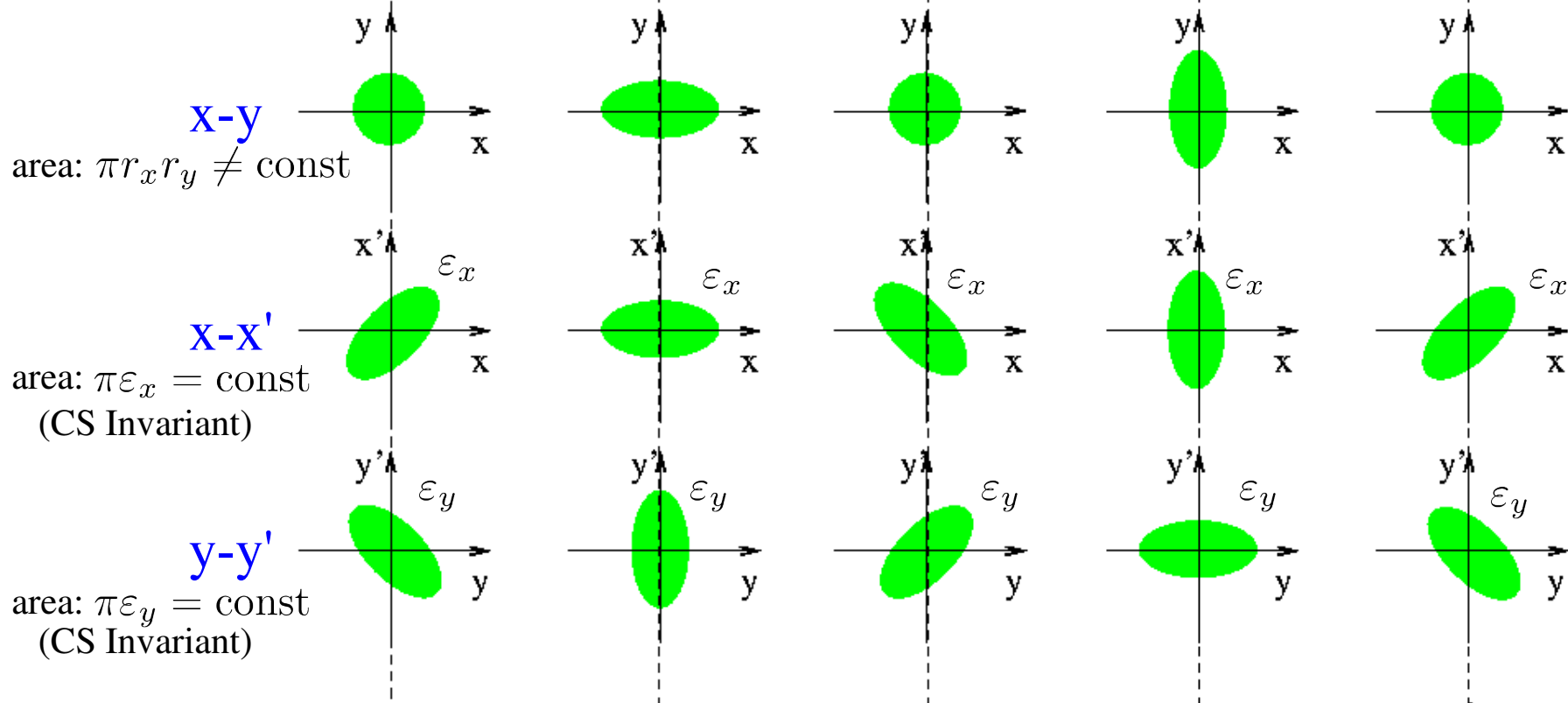


Symmetries of a matched beam are interpreted in terms of a local rms equivalent KV beam and moments/projections of the KV distribution

[see: S.M. Lund, lectures on **Transverse Equilibrium Distributions**]



Projection



S6: Envelope Perturbations:

An extensive review article is available that both reviews/extends many aspects of envelope modes in periodic lattices covered in **S6-S8**: see S.M. Lund and B. Bukh, PRSTAB 024801 (2004) [henceforth denoted: PRSTAB Review]

In the envelope equations take:

Envelope Perturbations:

$$\begin{aligned}
 r_x(s) &= r_{xm}(s) + \delta r_x(s) \\
 r_y(s) &= r_{ym}(s) + \delta r_y(s)
 \end{aligned}$$

Matched Mismatch
Envelope Perturbations

Driving Perturbations:

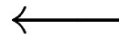
$$\begin{aligned}
 \kappa_x(s) &\rightarrow \kappa_x(s) + \delta\kappa_x(s) && \text{Focus} \\
 \kappa_y(s) &\rightarrow \kappa_y(s) + \delta\kappa_y(s) && \\
 Q &\rightarrow Q + \delta Q(s) && \text{Perveance} \\
 \varepsilon_x &\rightarrow \varepsilon_x + \delta\varepsilon_x(s) && \text{Emittance} \\
 \varepsilon_y &\rightarrow \varepsilon_y + \delta\varepsilon_y(s) &&
 \end{aligned}$$

$$r_{xm}(s + L_p) = r_{xm}(s) \quad r_{xm}(s) > 0$$

$$r_{ym}(s + L_p) = r_{ym}(s) \quad r_{ym}(s) > 0$$

$$r_{xm}(s) \gg |\delta r_{xm}(s)|$$

$$r_{ym}(s) \gg |\delta r_{ym}(s)|$$



Amplitudes defined in terms of producing small envelope perturbations

- ◆ Driving perturbations and distribution errors generate/pump envelope perturbations
 - Arise from many sources: focusing errors, lost particles, emittance growth,

The **matched solution** satisfies:

- ◆ Add subscript m to denote matched to distinguish from other solutions

$$\begin{aligned} r_x &\rightarrow r_{xm} && \text{For matched beam envelope} \\ r_y &\rightarrow r_{ym} && \text{with periodicity of lattice} \end{aligned}$$

$$r_{xm}''(s) + \kappa_x(s)r_{xm}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} = 0$$

$$r_{ym}''(s) + \kappa_y(s)r_{ym}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_y^2}{r_{ym}^3(s)} = 0$$

$$r_{xm}(s + L_p) = r_{xm}(s) \quad r_{xm}(s) > 0$$

$$r_{ym}(s + L_p) = r_{ym}(s) \quad r_{ym}(s) > 0$$

Linearized Perturbed Envelope Equations:

- ◆ Neglect all terms of order δ^2 and higher: $(\delta r_x)^2$, $\delta r_x \delta r_y$, $\delta Q \delta r_x$, \dots

$$\begin{aligned}\delta r_x'' + \kappa_x \delta r_x + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4} \delta r_x \\ = -r_{xm} \delta \kappa_x + \frac{2}{r_{xm} + r_{ym}} \delta Q + \frac{2\varepsilon_x^2}{r_{xm}^3} \delta \varepsilon_x\end{aligned}$$
$$\begin{aligned}\delta r_y'' + \kappa_y \delta r_y + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4} \delta r_y \\ = -r_{ym} \delta \kappa_y + \frac{2}{r_{xm} + r_{ym}} \delta Q + \frac{2\varepsilon_y^2}{r_{ym}^3} \delta \varepsilon_y\end{aligned}$$

Homogeneous Equations:

- ◆ Linearized envelope equations with driving terms set to zero

$$\begin{aligned}\delta r_x'' + \kappa_x \delta r_x + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4} \delta r_x = 0 \\ \delta r_y'' + \kappa_y \delta r_y + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4} \delta r_y = 0\end{aligned}$$

Martix Form of the Linearized Perturbed Envelope Equations:

$$\frac{d}{ds} \delta \mathbf{R} + \mathbf{K} \cdot \delta \mathbf{R} = \delta \mathbf{P}$$

$$\delta \mathbf{R} \equiv \begin{pmatrix} \delta r_x \\ \delta r'_x \\ \delta r_y \\ \delta r'_y \end{pmatrix} \quad \text{Coordinate vector}$$

$$\mathbf{K} \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ k_{xm} & 0 & k_{0m} & 0 \\ 0 & 0 & 0 & -1 \\ k_{0m} & 0 & k_{ym} & 0 \end{pmatrix}$$

Coefficient matrix Has periodicity of the lattice period

$$k_{0m} = \frac{2Q}{(r_{xm} + r_{ym})^2}$$

$$k_{jm} = \kappa_j + 3 \frac{\varepsilon_j^2}{r_{jm}^4} + k_{0m} \quad j = x, y$$

$$\delta \mathbf{P} \equiv \begin{pmatrix} 0 \\ -\delta \kappa_x + 2 \frac{\delta Q}{r_{xm} + r_{ym}} + 2 \frac{\varepsilon_x \delta \varepsilon_x}{r_{xm}^3} \\ 0 \\ -\delta \kappa_y + 2 \frac{\delta Q}{r_{xm} + r_{ym}} + 2 \frac{\varepsilon_y \delta \varepsilon_y}{r_{ym}^3} \end{pmatrix}$$

Driving perturbation vector

Expand solution into **homogeneous** and **particular** parts:

$$\delta \mathbf{R} = \delta \mathbf{R}_h + \delta \mathbf{R}_p$$

$\delta \mathbf{R}_h =$ **homogeneous** solution

$\delta \mathbf{R}_p =$ **particular** solution

$$\frac{d}{ds} \delta \mathbf{R}_h + \mathbf{K} \cdot \delta \mathbf{R}_h = 0$$

$$\frac{d}{ds} \delta \mathbf{R}_p + \mathbf{K} \cdot \delta \mathbf{R}_p = \delta \mathbf{P}$$

Homogeneous Solution: Normal Modes

- ◆ Describes normal mode oscillations
- ◆ Original analysis by Struckmeier and Reiser [Part. Accel. **14**, 227 (1984)]

Particular Solution: Driven Modes

- ◆ Describes action of driving terms
- ◆ Characterize in terms of projections on homogeneous response (on normal modes)

Homogeneous solution expressible as a map:

$$\begin{aligned}\delta\mathbf{R}(s) &= \mathbf{M}_e(s|s_i) \cdot \delta\mathbf{R}(s_i) \\ \delta\mathbf{R}(s) &= (\delta r_x, \delta r'_x, \delta r_y, \delta r'_y) \\ \mathbf{M}_e(s|s_i) &= 4 \times 4 \text{ transfer map}\end{aligned}$$

Now 4x4 system, but analogous to the 2x2 analysis of Hill's equation via transfer matrices: see S.M. Lund lectures on **Transverse Particle Equations**

Eigenvalues and eigenvectors of map through one period characterize normal modes and stability properties:

$$\mathbf{M}_e(s_i + L_p|s_i) \cdot \mathbf{E}_n(s_i) = \lambda_n \mathbf{E}_n(s_i)$$

Stability

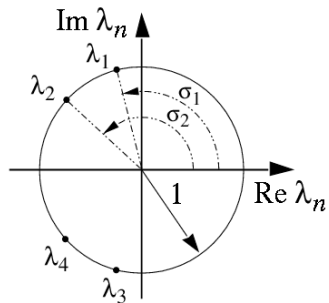
$$\lambda_n = \gamma_n e^{i\sigma_n} \quad \begin{array}{l} \sigma_n \rightarrow \text{mode phase advance (real)} \\ \gamma_n \rightarrow \text{mode growth/damp factor (real)} \end{array}$$

Mode Expansion/Launching

$$\begin{aligned}\delta\mathbf{R}(s_i) &= \sum_{n=1}^4 \alpha_n \mathbf{E}_n(s_i) \\ \alpha_n &= \text{const (complex)}\end{aligned}$$

Eigenvalue/Eigenvector Symmetry Classes:

a) Stable



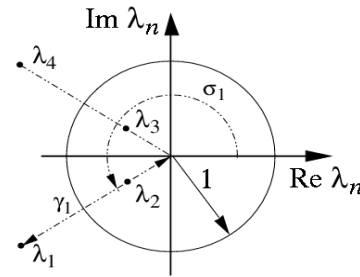
Eigenvalues

$$\begin{aligned}\lambda_1 &= e^{i\sigma_1} \\ \lambda_2 &= e^{i\sigma_2} \\ \lambda_3 &= 1/\lambda_1 = \lambda_1^* = e^{-i\sigma_1} \\ \lambda_4 &= 1/\lambda_2 = \lambda_2^* = e^{-i\sigma_2}\end{aligned}$$

Eigenvectors

$$\begin{aligned}\vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 &= \vec{E}_1^* \\ \vec{E}_4 &= \vec{E}_2^*\end{aligned}$$

b) Unstable, Confluent Resonance



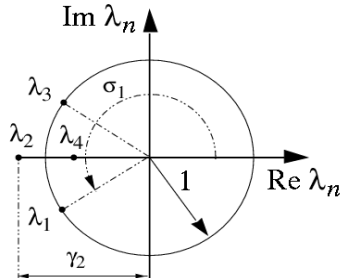
Eigenvalues

$$\begin{aligned}\lambda_1 &= \gamma_1 e^{i\sigma_1} \\ \lambda_2 &= 1/\lambda_1^* = (1/\gamma_1) e^{i\sigma_1} \\ \lambda_3 &= 1/\lambda_1 = (1/\gamma_1) e^{-i\sigma_1} \\ \lambda_4 &= \lambda_1^* = \gamma_1 e^{-i\sigma_1}\end{aligned}$$

Eigenvectors

$$\begin{aligned}\vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 &= \vec{E}_2^* \\ \vec{E}_4 &= \vec{E}_1^*\end{aligned}$$

c) Unstable, Lattice Resonance



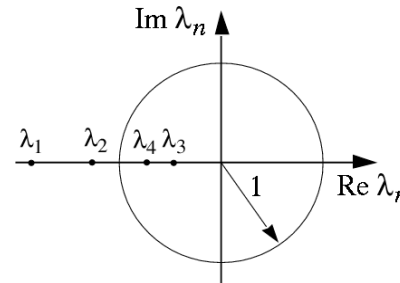
Eigenvalues

$$\begin{aligned}\lambda_1 &= e^{i\pi} \\ \lambda_2 &= \gamma_2 e^{i\pi} \\ \lambda_3 &= \lambda_1^* = e^{-i\pi} \\ \lambda_4 &= 1/\lambda_2 = (1/\gamma_2) e^{i\pi}\end{aligned}$$

Eigenvectors

$$\begin{aligned}\vec{E}_1 \\ \vec{E}_2 \text{ (real)} \\ \vec{E}_3 &= \vec{E}_1^* \\ \vec{E}_4 \text{ (real)}\end{aligned}$$

d) Unstable, Double Lattice Resonance



Eigenvalues

$$\begin{aligned}\lambda_1 &= \gamma_1 e^{i\pi} \\ \lambda_2 &= \gamma_2 e^{i\pi} \\ \lambda_3 &= 1/\lambda_1 = (1/\gamma_1) e^{i\pi} \\ \lambda_4 &= 1/\lambda_2 = (1/\gamma_2) e^{i\pi}\end{aligned}$$

Eigenvectors

$$\begin{aligned}\vec{E}_1 \text{ (real)} \\ \vec{E}_2 \text{ (real)} \\ \vec{E}_3 \text{ (real)} \\ \vec{E}_4 \text{ (real)}\end{aligned}$$

Symmetry classes of eigenvalues/eigenvectors:

- ◆ Determine normal mode symmetries
- ◆ See A. Dragt, Lectures on Nonlinear Orbit Dynamics, in Physics of High Energy Particle Accelerators, (AIP Conf. Proc. No. 87, 1982, p. 147)
- ◆ Envelope mode symmetries discussed fully in PRSTAB review
- ◆ Caution: Textbook by Reiser makes errors in mode symmetries and mislabels/identifies dispersion characteristics

Pure mode launching conditions:

Launching conditions for distinct normal modes corresponding to the eigenvalue classes illustrated:

A_ℓ = mode amplitude (real)

ℓ = mode index

ψ_ℓ = mode launch phase (real)

C.C. = complex conjugate

Case	Mode	Launching Condition	Lattice Period Advance
(a) Stable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
	2 - Stable Osc.	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2} \mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e \delta\mathbf{R}_2(\psi_2) = \delta\mathbf{R}_2(\psi_2 + \sigma_2)$
(b) Unstable Confluent Res.	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \gamma_1 \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
	2 - Exp. Damping	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2} \mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e \delta\mathbf{R}_2(\psi_2) = (1/\gamma_1) \delta\mathbf{R}_2(\psi_2 + \sigma_1)$
(c) Unstable Lattice Res.	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2 \mathbf{E}_2$	$\mathbf{M}_e \delta\mathbf{R}_2 = -\gamma_2 \delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3 \mathbf{E}_4$	$\mathbf{M}_e \delta\mathbf{R}_3 = -(1/\gamma_2) \delta\mathbf{R}_3$
(d) Unstable Double Lattice Resonance	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 \mathbf{E}_1$	$\mathbf{M}_e \delta\mathbf{R}_1 = -\gamma_1 \delta\mathbf{R}_1$
	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2 \mathbf{E}_2$	$\mathbf{M}_e \delta\mathbf{R}_2 = -\gamma_2 \delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3 \mathbf{E}_3$	$\mathbf{M}_e \delta\mathbf{R}_3 = -(1/\gamma_1) \delta\mathbf{R}_3$
	4 - Exp. Damping	$\delta\mathbf{R}_4 = A_4 \mathbf{E}_4$	$\mathbf{M}_e \delta\mathbf{R}_4 = -(1/\gamma_2) \delta\mathbf{R}_4$

$$\delta\mathbf{R}_\ell \equiv \delta\mathbf{R}_\ell(s_i) \quad \mathbf{E}_\ell \equiv \mathbf{E}_\ell(s_i) \quad \mathbf{M}_e \equiv \mathbf{M}_e(s_i + L_p | s_i)$$

$$\delta\mathbf{R}(s) = \begin{cases} A_1[\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2[\mathbf{E}_2(s)e^{i\psi_2(s)} + \mathbf{E}_2^*(s)e^{-i\psi_2(s)}], & \text{cases (a) and (b)} \\ A_1[\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2\mathbf{E}_2(s) + A_3\mathbf{E}_4(s), & \text{case (c)} \\ A_1\mathbf{E}_1(s) + A_2\mathbf{E}_2(s) + A_3\mathbf{E}_3(s) + A_4\mathbf{E}_4(s), & \text{case (d)} \end{cases}$$

Decoupled Modes

In a **continuous** or **periodic solenoidal** focusing channel

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

with a round matched-beam solution

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const}$$

$$r_{xm}(s) = r_{ym}(s) \equiv r_m(s)$$

envelope perturbations are simply decoupled with:

$$\text{Breathing Mode:} \quad \delta r_+ \equiv \frac{\delta r_x + \delta r_y}{2}$$

$$\text{Quadrupole Mode:} \quad \delta r_- \equiv \frac{\delta r_x - \delta r_y}{2}$$

The resulting decoupled envelope equations are:

$$\text{Breathing Mode:} \quad \delta r_+'' + \kappa \delta r_+ + \frac{2Q}{r_m^2} \delta r_+ + \frac{3\varepsilon^2}{r_m^4} \delta r_+ = -r_m \left(\frac{\delta \kappa_x + \delta \kappa_y}{2} \right) + \frac{1}{r_m} \delta Q + \frac{2\varepsilon^2}{r_m^3} \left(\frac{\delta \varepsilon_x + \delta \varepsilon_y}{2} \right)$$

$$\text{Quadrupole Mode:} \quad \delta r_-'' + \kappa \delta r_- + \frac{3\varepsilon^2}{r_m^4} \delta r_- = -r_m \left(\frac{\delta \kappa_x - \delta \kappa_y}{2} \right) + \frac{2\varepsilon^2}{r_m^3} \left(\frac{\delta \varepsilon_x - \delta \varepsilon_y}{2} \right)$$

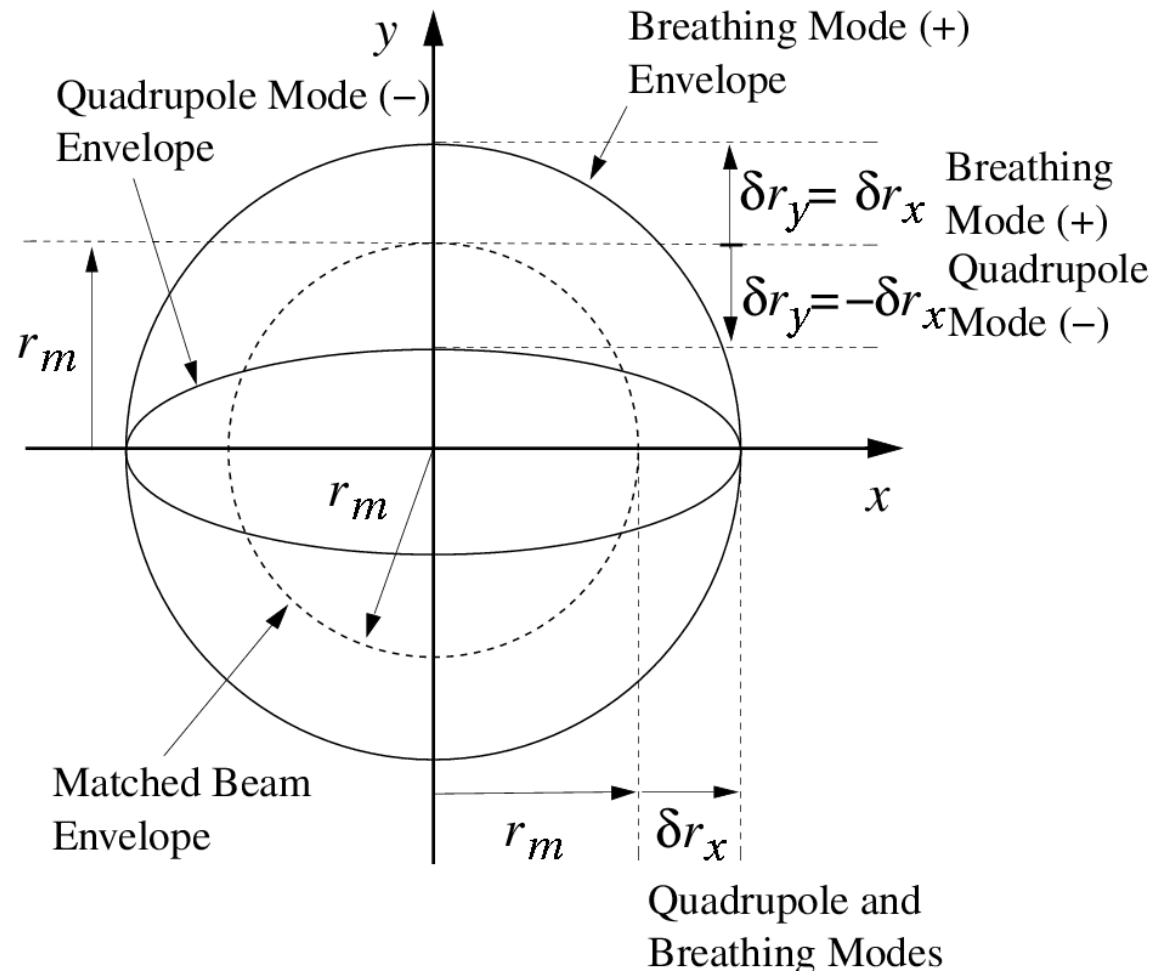
Graphical interpretation of mode symmetries:

Breathing Mode:

$$\delta r_+ = \frac{\delta r_x + \delta r_y}{2}$$

Quadrupole Mode:

$$\delta r_- = \frac{\delta r_x - \delta r_y}{2}$$



Decoupled Mode Properties:

Space charge terms $\sim Q$ only directly expressed in equation for $\delta r_+(s)$

- ◆ Indirectly present in both equations from matched envelope $r_m(s)$

Homogeneous Solution:

- ◆ Restoring term for $\delta r_+(s)$ larger than for $\delta r_-(s)$
 - Breathing mode should oscillate faster than the quadrupole mode

Particular Solution:

- ◆ Misbalances in focusing and emittance driving terms can project onto either mode
 - nonzero perturbed $\kappa_x(s) + \kappa_y(s)$ and $\varepsilon_x(s) + \varepsilon_y(s)$
project onto breathing mode
 - nonzero perturbed $\kappa_x(s) - \kappa_y(s)$ and $\varepsilon_x(s) - \varepsilon_y(s)$
project onto quadrupole mode
- ◆ Perveance driving perturbations project *only* on breathing mode

Previous symmetry classes greatly reduce for decoupled modes:

Previous homogeneous 4x4 solution map:

$$\delta\mathbf{R}(s) = \mathbf{M}_e(s|s_i) \cdot \delta\mathbf{R}(s_i)$$

$$\delta\mathbf{R}(s) = (\delta r_x, \delta r'_x, \delta r_y, \delta r'_y)$$

$$\mathbf{M}_e(s|s_i) = 4 \times 4 \text{ transfer map}$$

reduces to two independent 2x2 maps with greatly simplified symmetries:

$$\delta\mathbf{R} \equiv (\delta r_+, \delta r'_+, \delta r_-, \delta r'_-)$$

$$\mathbf{M}_e(s_i + L_p|s_i) = \begin{bmatrix} \mathbf{M}_+(s_i + L_p|s_i) & 0 \\ 0 & \mathbf{M}_-(s_i + L_p|s_i) \end{bmatrix}$$

with corresponding eigenvalue problems:

$$\mathbf{M}_\pm(s_i + L_p|s_i) \cdot \mathbf{E}_n(s_i) = \lambda_\pm \mathbf{E}_n(s_i)$$

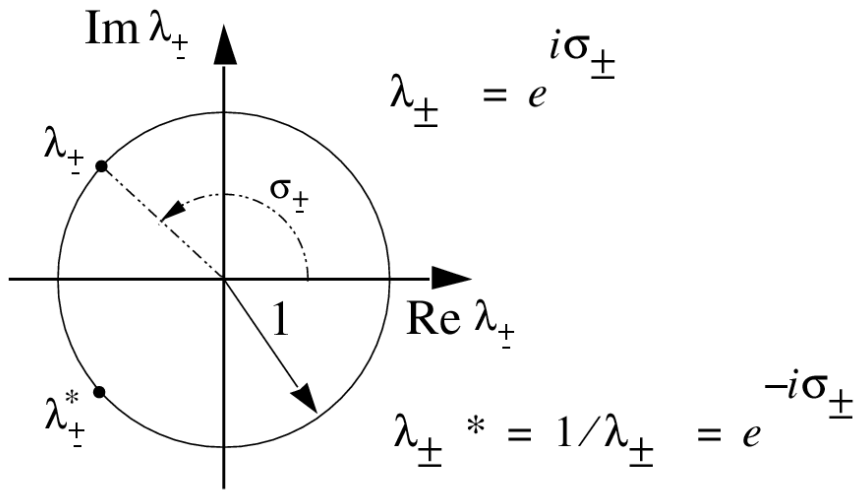
Many familiar results from analysis of Hills equation (see: S.M. Lund lectures on **Transverse Particle Equations**) can be immediately applied to the decoupled case, for example:

$$\frac{1}{2} |\text{Tr } \mathbf{M}_\pm(s_i + L_p|s_i)| < 1 \quad \longrightarrow \quad \text{mode stability}$$

Eigenvalue symmetries and launching conditions simplify for decoupled modes

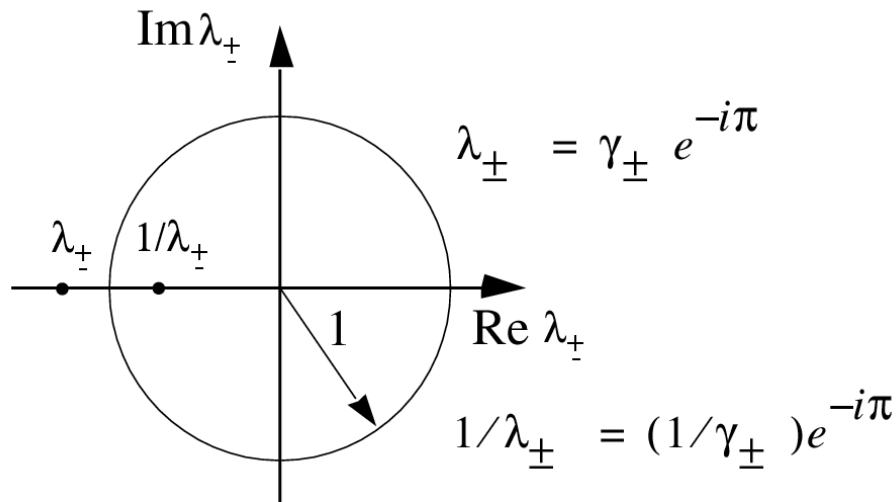
Eigenvalue Symmetry 1:

Stable

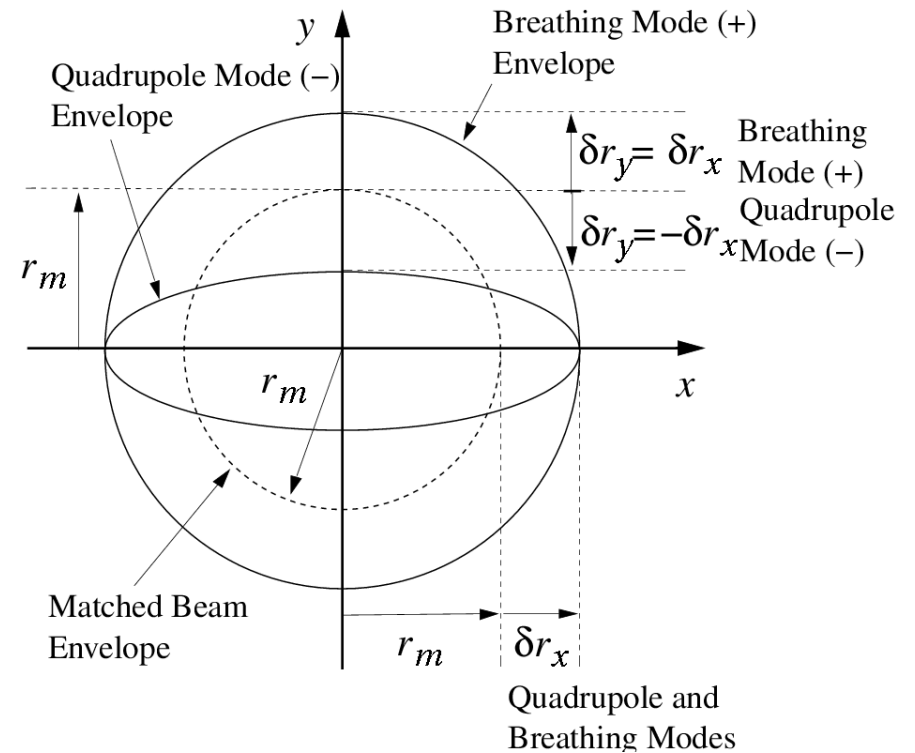


Eigenvalue Symmetry 2:

Unstable, Lattice Resonance



Launching Condition / Projections



General Mode Limits

Using phase-amplitude analysis can show for any linear focusing lattice:

- 1) Phase advance of any normal mode satisfies the zero space-charge limit:

$$\lim_{Q \rightarrow 0} \sigma_\ell = 2\sigma_0$$

- 2) Pure normal modes (not driven) evolve with a quadratic phase-space (Courant-Snyder) invariant in the normal coordinates of the mode

Simply expressed for decoupled modes with $\kappa_x = \kappa_y$, $\varepsilon_x = \varepsilon_y$

$$\left[\frac{\delta r_\pm(s)}{w_\pm(s)} \right]^2 + [w'_\pm(s)\delta r_\pm(s) - w_\pm(s)\delta r'_\pm(s)]^2 = \text{const}$$

where

$$w_+'' + \kappa w_+ + \frac{2Q}{r_m^2} w_+ + \frac{3\varepsilon^2}{r_m^4} w_+ - \frac{1}{w_+^3} = 0$$

$$w_-'' + \kappa w_- + \frac{3\varepsilon^2}{r_m^4} w_- - \frac{1}{w_-^3} = 0$$

$$w_\pm(s + L_p) = w_\pm(s)$$

Analogous results for coupled modes [See Edwards and Teng, IEEE Trans Nuc. Sci. **20**, 885 (1973)]

- ◆ More complex expression due to coupling

S7: Envelope Modes in Continuous Focusing

Focusing: $\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2 = \text{const}$

Matched beam:
symmetric beam: $\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$
 $r_{xm}(s) = r_{ym}(s) = r_m = \text{const}$

matched envelope: $k_{\beta 0}^2 r_m - \frac{Q}{r_m} - \frac{\varepsilon^2}{r_m^3} = 0$

depressed phase advance: $\sigma = \sqrt{\sigma_0^2 - \frac{Q}{(r_m/L_p)^2}} = \frac{\varepsilon L_p}{r_m^2}$

one parameter needed for scaled solution:
Decoupled Modes: $\frac{k_{\beta 0}^2 \varepsilon^2}{Q^2} = \frac{\sigma_0^2 \varepsilon^2}{Q^2 L_p^2} = \frac{(\sigma/\sigma_0)^2}{[1 - (\sigma/\sigma_0)^2]^2}$

$$\delta r_{\pm}(s) = \frac{\delta r_x(s) \pm \delta r_y(s)}{2}$$

Envelope equations of motion become:

$$L_p^2 \frac{d^2}{ds^2} \left(\frac{\delta r_+}{r_m} \right) + \sigma_+^2 \left(\frac{\delta r_+}{r_m} \right) = -\frac{\sigma_0^2}{2} \left(\frac{\delta \kappa_x}{k_{\beta 0}^2} + \frac{\delta \kappa_y}{k_{\beta 0}^2} \right) + (\sigma_0^2 - \sigma^2) \frac{\delta Q}{Q} + \sigma^2 \left(\frac{\delta \varepsilon_x}{\varepsilon} + \frac{\delta \varepsilon_y}{\varepsilon} \right)$$

$$L_p^2 \frac{d^2}{ds^2} \left(\frac{\delta r_-}{r_m} \right) + \sigma_-^2 \left(\frac{\delta r_-}{r_m} \right) = -\frac{\sigma_0^2}{2} \left(\frac{\delta \kappa_x}{k_{\beta 0}^2} - \frac{\delta \kappa_y}{k_{\beta 0}^2} \right) + \sigma^2 \left(\frac{\delta \varepsilon_x}{\varepsilon} - \frac{\delta \varepsilon_y}{\varepsilon} \right)$$

$$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2} \quad \text{“breathing”} \quad \text{mode phase advance}$$

$$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2} \quad \text{“quadrupole”} \quad \text{mode phase advance}$$

Homogeneous equations for normal modes:

$$\frac{d^2}{ds^2} \delta r_{\pm} + \left(\frac{\sigma_{\pm}}{L_p} \right)^2 \delta r_{\pm} = 0$$

♦ Simple harmonic oscillator equation

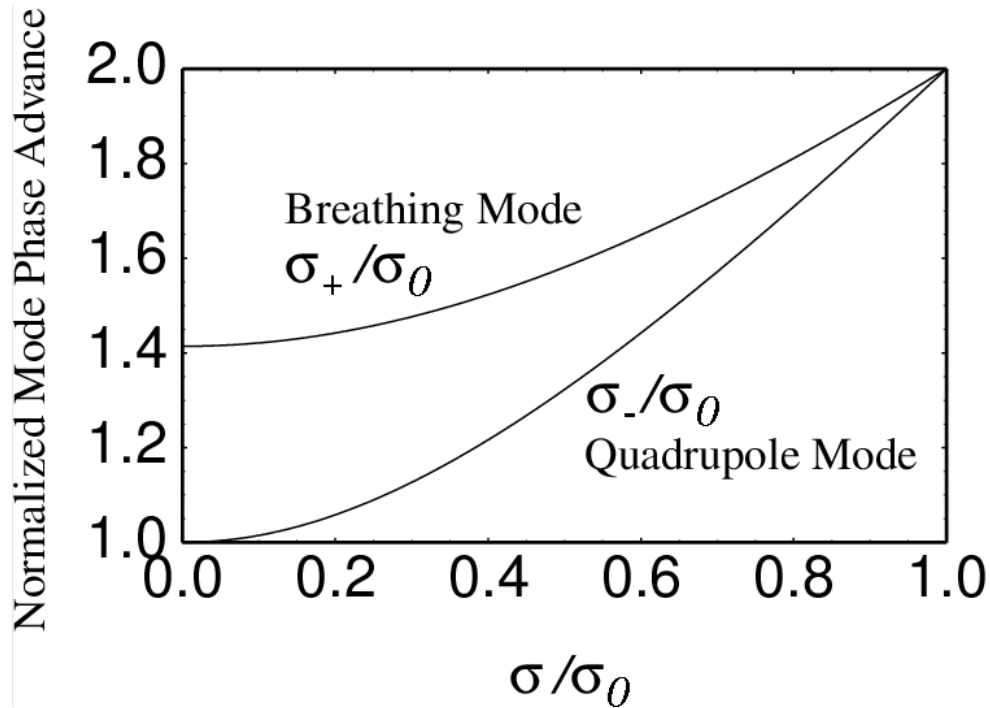
Homogeneous Solution (normal modes):

$$\delta r_{\pm}(s) = \delta r_{\pm}(s_i) \cos \left(\sigma_{\pm} \frac{s - s_i}{L_p} \right) + \frac{\delta r'_{\pm}(s_i)}{\sigma_{\pm}/L_p} \sin \left(\sigma_{\pm} \frac{s - s_i}{L_p} \right)$$

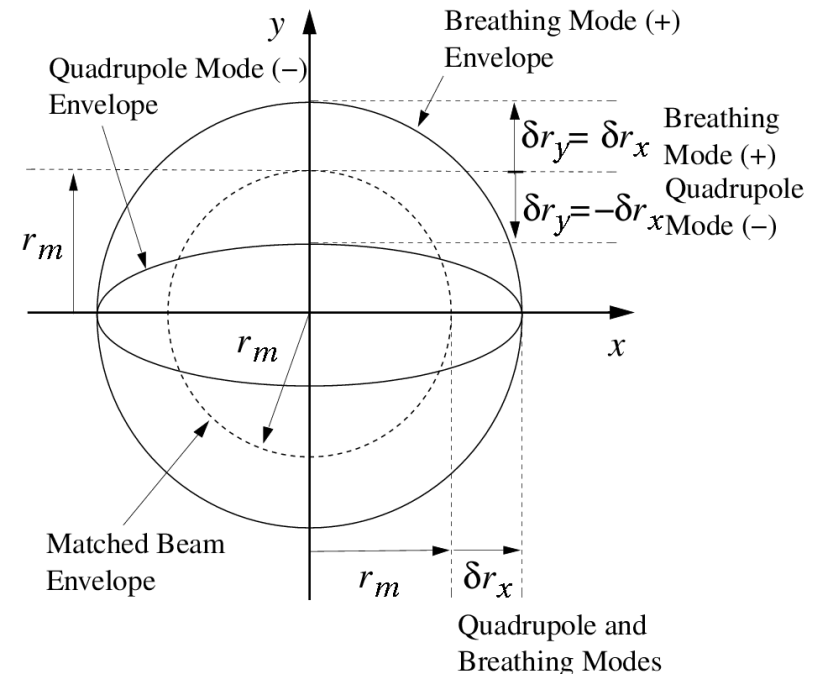
$\delta r_{\pm}(s_i), \delta r'_{\pm}(s_i)$ mode initial conditions

Properties of continuous focusing homogeneous solution: Normal Modes

Mode Phase Advances



Mode Projections



Breathing Mode:
$$\delta r_+ \equiv \frac{\delta r_x + \delta r_y}{2}$$

Quadrupole Mode:
$$\delta r_- \equiv \frac{\delta r_x - \delta r_y}{2}$$

Particular Solution (driving perturbations):

Green's function form of solution derived using projections onto normal modes
see PRSTAB Review

$$\frac{\delta r_{\pm}(s)}{r_m} = \frac{1}{L_p^2} \int_{s_i}^s d\tilde{s} G_{\pm}(s, \tilde{s}) \delta p_{\pm}(\tilde{s})$$

$$\delta p_{+}(s) = -\frac{\sigma_0^2}{2} \left[\frac{\delta \kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta \kappa_y(s)}{k_{\beta 0}^2} \right] + (\sigma_0^2 - \sigma^2) \frac{\delta Q(s)}{Q} + \sigma^2 \left[\frac{\delta \varepsilon_x(s)}{\varepsilon} + \frac{\delta \varepsilon_y(s)}{\varepsilon} \right]$$

$$\delta p_{-}(s) = -\frac{\sigma_0^2}{2} \left[\frac{\delta \kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta \kappa_y(s)}{k_{\beta 0}^2} \right] + \sigma^2 \left[\frac{\delta \varepsilon_x(s)}{\varepsilon} - \frac{\delta \varepsilon_y(s)}{\varepsilon} \right]$$

$$G_{\pm}(s, \tilde{s}) = \frac{1}{\sigma_{\pm}/L_p} \sin \left(\sigma_{\pm} \frac{s - \tilde{s}}{L_p} \right)$$

Green's function solution is *fully general*. Insight gained from simplified solutions for specific classes of driving perturbations:

- ◆ **Adiabatic**
- ◆ **Sudden** covered here
- ◆ **Ramped**
- ◆ **Harmonic** covered in PRSTAB Review article

Continuous Focusing – adiabatic particular solution

For driving perturbations $\delta p_+(s)$ and $\delta p_-(s)$ slow on quadrupole mode (slower mode) wavelength $\sim 2\pi L_p/\sigma_-$ the solution is:

$$\begin{aligned} \frac{\delta r_+(s)}{r_m} &= \frac{\delta p_+(s)}{\sigma_+^2} \\ &= - \left[\frac{1}{2} \frac{1}{1 + (\sigma/\sigma_0)^2} \right] \frac{1}{2} \left(\frac{\delta \kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta \kappa_y(s)}{k_{\beta 0}^2} \right) + \left[\frac{1}{2} \frac{1 - (\sigma/\sigma_0)^2}{1 + (\sigma/\sigma_0)^2} \right] \frac{\delta Q(s)}{Q} \\ &\quad + \left[\frac{(\sigma/\sigma_0)^2}{1 + (\sigma/\sigma_0)^2} \right] \frac{1}{2} \left(\frac{\delta \varepsilon_x(s)}{\varepsilon} + \frac{\delta \varepsilon_y(s)}{\varepsilon} \right), \end{aligned}$$

Focusing
Perveance

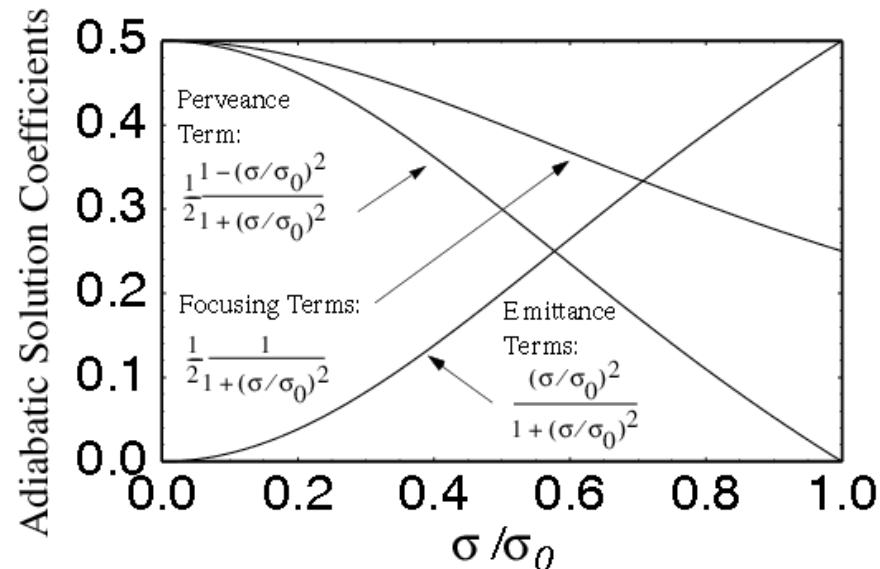
$$\begin{aligned} \frac{\delta r_-(s)}{r_m} &= \frac{\delta p_-(s)}{\sigma_-^2} \\ &= - \left[\frac{1}{1 + 3(\sigma/\sigma_0)^2} \right] \frac{1}{2} \left(\frac{\delta \kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta \kappa_y(s)}{k_{\beta 0}^2} \right) \\ &\quad + \left[\frac{2(\sigma/\sigma_0)^2}{1 + 3(\sigma/\sigma_0)^2} \right] \frac{1}{2} \left(\frac{\delta \varepsilon_x(s)}{\varepsilon} - \frac{\delta \varepsilon_y(s)}{\varepsilon} \right). \end{aligned}$$

Emittance
Focusing
Coefficients of adiabatic terms in square brackets “[]”

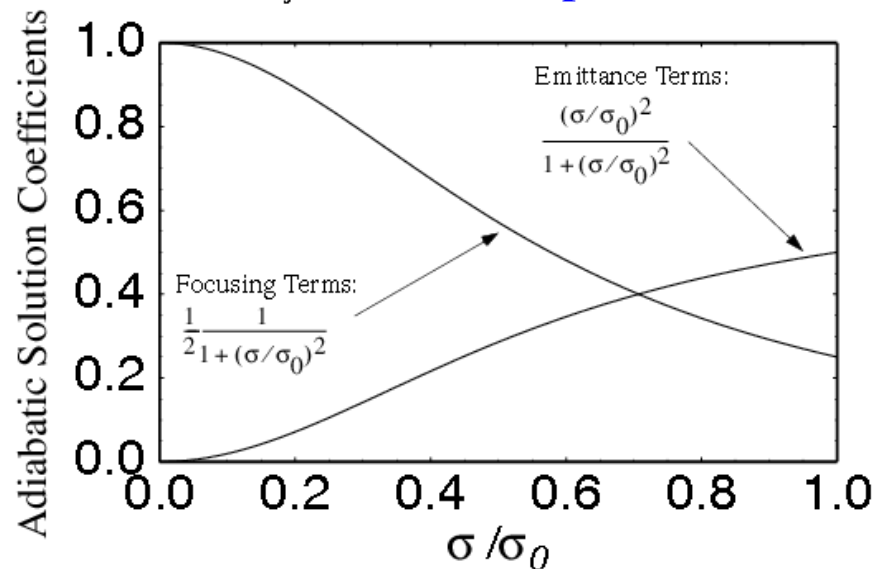
Emittance

Continuous Focusing – adiabatic solution coefficients

a) $\delta r_+ = (\delta r_x + \delta r_y)/2$ Breathing Mode Projection



b) $\delta r_- = (\delta r_x - \delta r_y)/2$ Quadrupole Mode Projection



Relative strength of:

- ◆ Space-Charge (Perveance)
- ◆ Applied Focusing
- ◆ Emittance

terms vary with space-charge depression (σ/σ_0) for both breathing and quadrupole mode projections

Plots allow one to read off the relative importance of various contributions to beam mismatch as a function of space-charge strength

Continuous Focusing – sudden particular solution

For step function driving perturbations of form:

$$\delta p_{\pm}(s) = \widehat{\delta p}_{\pm} \Theta(s - s_p)$$

$s = s_p =$ axial coordinate
perturbation applied

Hat quantities
are constant
amplitudes

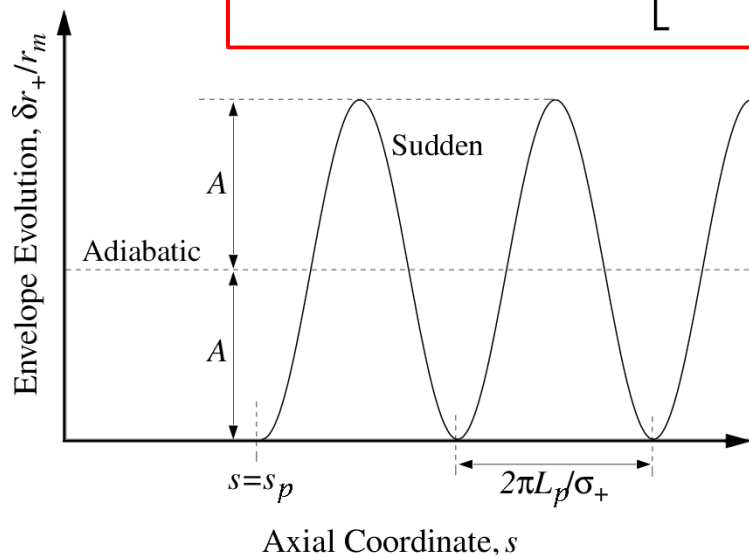
with amplitudes:

$$\widehat{\delta p}_+ = -\frac{\sigma_0^2}{2} \left[\frac{\widehat{\delta \kappa}_x}{k_{\beta 0}^2} + \frac{\widehat{\delta \kappa}_y}{k_{\beta 0}^2} \right] + (\sigma_0^2 - \sigma^2) \frac{\widehat{\delta Q}}{Q} + \sigma^2 \left[\frac{\widehat{\delta \varepsilon}_x}{\varepsilon} + \frac{\widehat{\delta \varepsilon}_y}{\varepsilon} \right] = \text{const}$$

$$\widehat{\delta p}_- = -\frac{\sigma_0^2}{2} \left[\frac{\widehat{\delta \kappa}_x}{k_{\beta 0}^2} - \frac{\widehat{\delta \kappa}_y}{k_{\beta 0}^2} \right] + \sigma^2 \left[\frac{\widehat{\delta \varepsilon}_x}{\varepsilon} - \frac{\widehat{\delta \varepsilon}_y}{\varepsilon} \right] = \text{const}$$

The solution is given by the substitution in the expression for the adiabatic solution:

$$\delta p_{\pm}(s) \rightarrow \widehat{\delta p}_{\pm} \left[1 - \cos \left(\sigma_{\pm} \frac{s - s_p}{L_p} \right) \right] \Theta(s - s_p)$$



2x Excursion

Adiabatic
Excursion

For the same amplitude of
total driving perturbations,
sudden perturbations result
in 2x the envelope
excursion that adiabatic
perturbations produce.

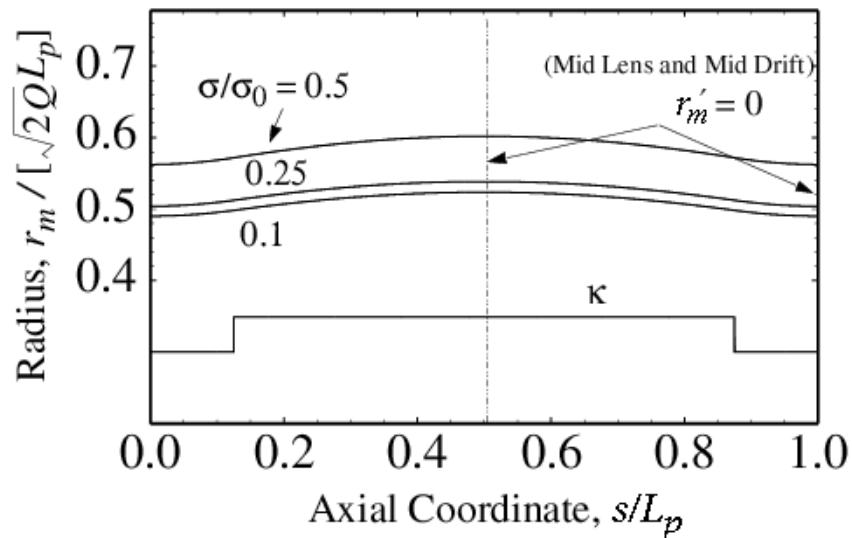
S8: Envelope Modes in Periodic Focusing Channels

Overview

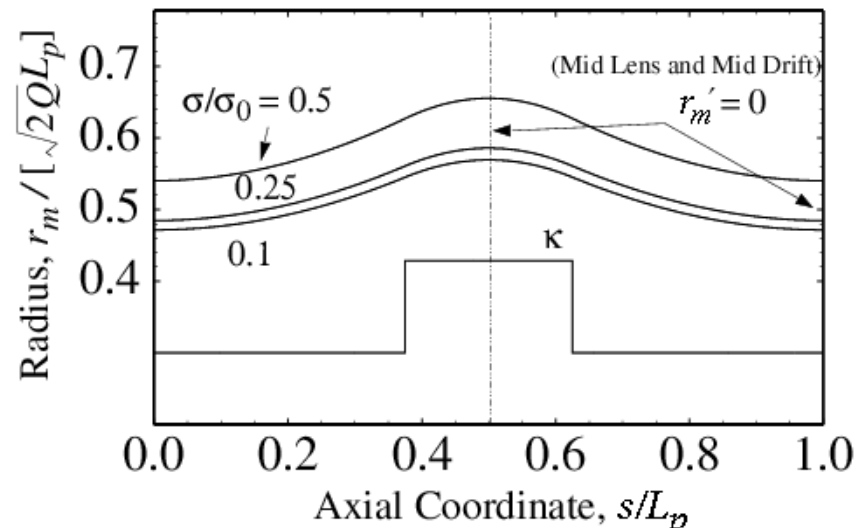
- ◆ Much more complicated the continuous limit results
 - Lattice can couple to oscillations and destabilize the system
 - Broad **parametric instability** bands can result
- ◆ Instability bands calculated will exclude wide ranges of parameter space from machine operation
 - Exclusion region depends on focusing type
 - Will find that alternating gradient quadrupole focusing tends to have more instability than high occupancy solenoidal focusing due to larger envelope flutter driving stronger, broader instability
- ◆ Results in this section are calculated numerically and summarized parametrically to illustrate the full range of mode characteristics
 - Results presented in terms of phase advances and normalized space-charge strength to allow broad applicability
 - Coupled 4x4 eigenvalue problem and mode symmetries identified in **S6** are solved numerically and analytical limits are verified
- ◆ More information on results presented can be found in the PRSTAB Review

Solenoidal Focusing – Matched Envelope Solution

a) $\sigma_\theta = 80^\circ$ and $\eta = 0.75$ **High Occupancy**



b) $\sigma_\theta = 80^\circ$ and $\eta = 0.25$ **Low Occupancy**



Focusing:

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

$$\kappa(s + L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

$$r_{xm}(s) = r_{ym}(s) = r_m(s)$$

$$r_m(s + L_p) = r_m(s)$$

Comments:

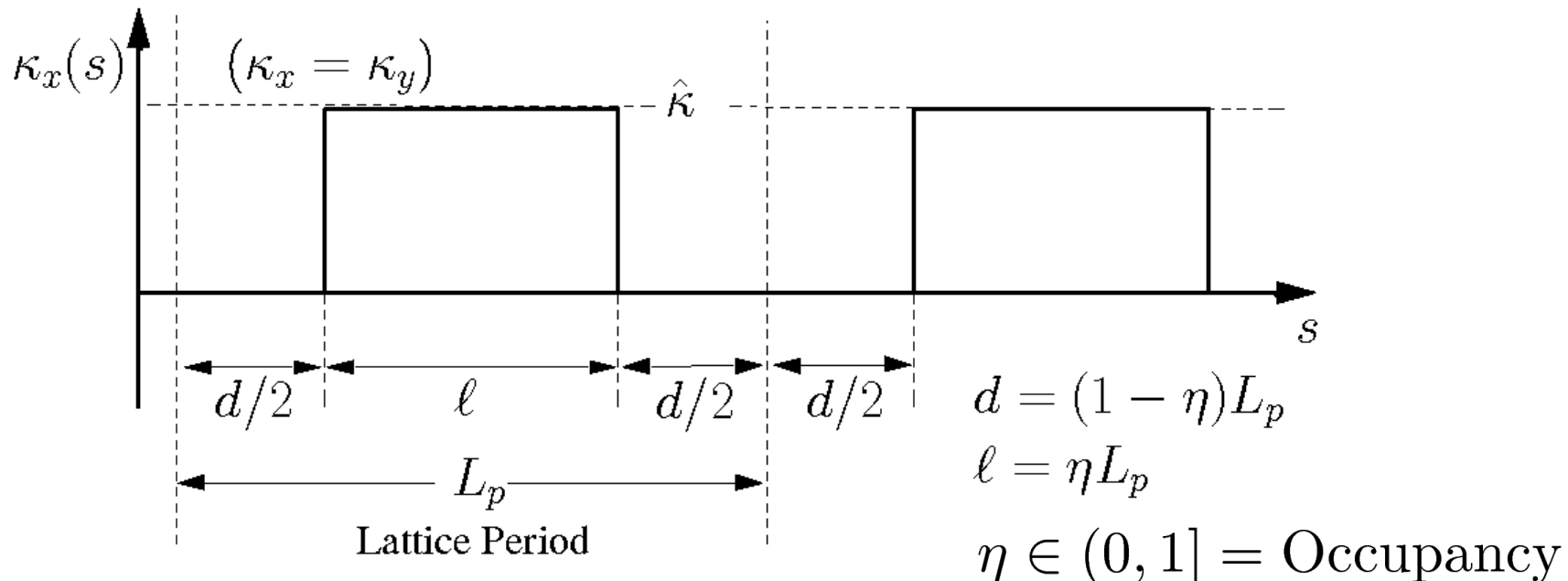
- ◆ Envelope flutter a strong function of occupancy η
- ◆ Space-charge expands envelope but does not strongly modify periodic flutter

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

- ♦ See: S.M. Lund, lectures on **Transverse Particle Equations**

Solenoidal Focusing - piecewise constant focusing lattice

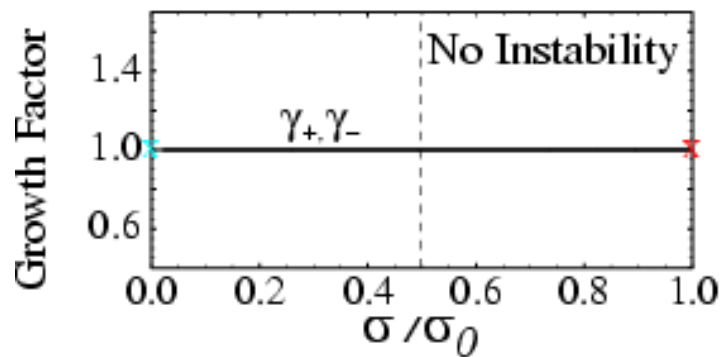
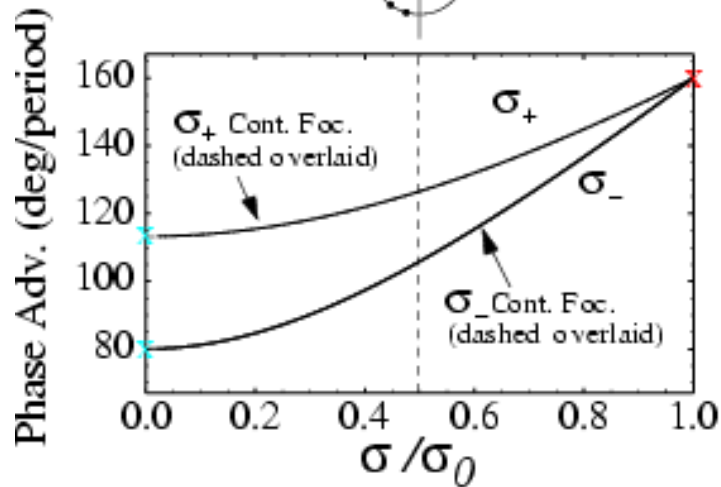
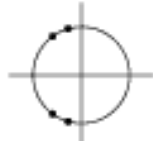
$$\cos \sigma_0 = \cos(2\Theta) - \frac{1 - \eta}{\eta} \Theta \sin(2\Theta) \quad \Theta \equiv \frac{\sqrt{\hat{\kappa}} L_p}{2}$$



Solenoidal Focusing – parametric plots of breathing and quadrupole envelope mode phase advances two values of underessed phase advance

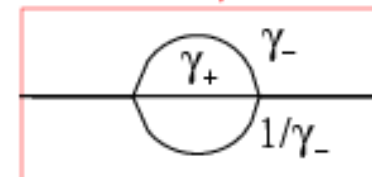
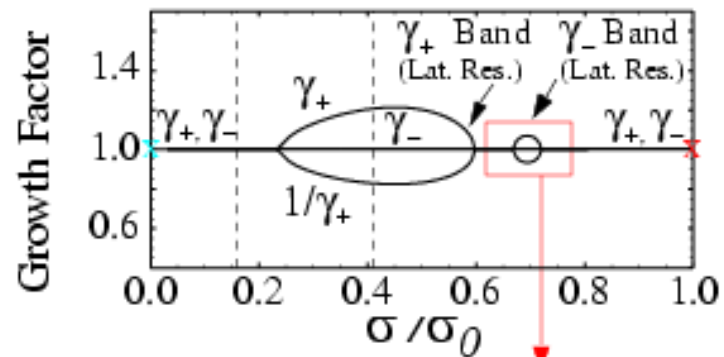
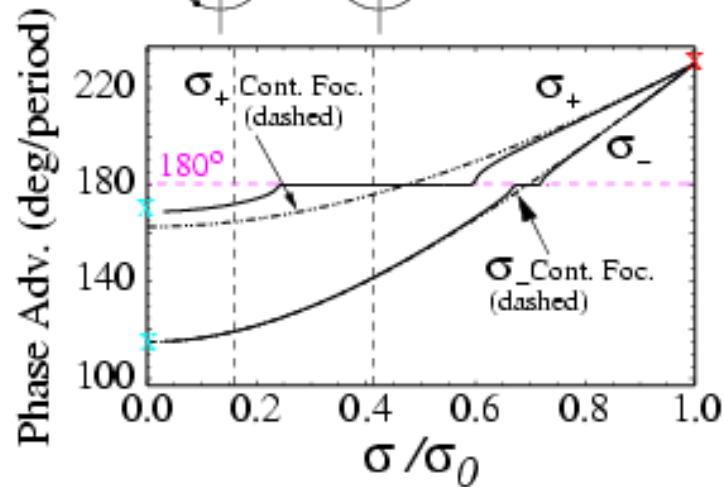
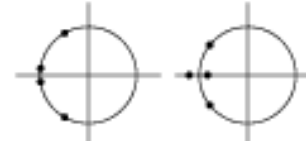
a) $\eta = 0.25, \sigma_0 = 80^\circ$

+: Stable
-: Stable



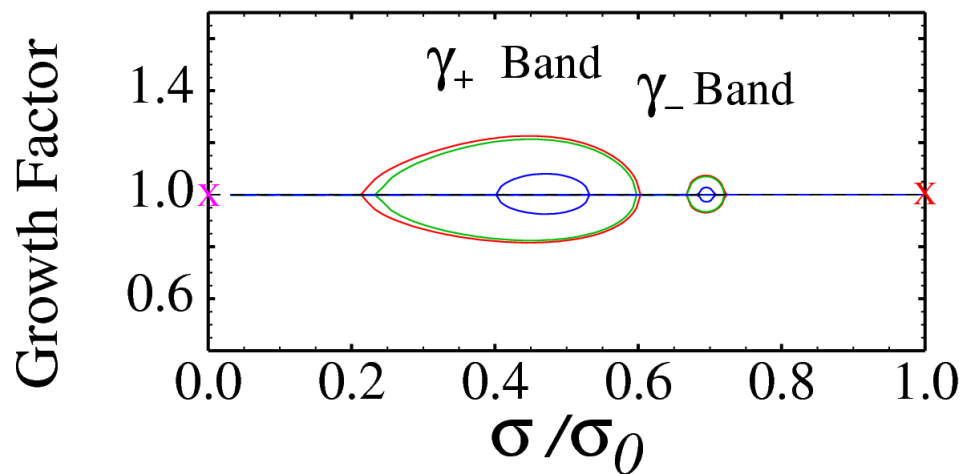
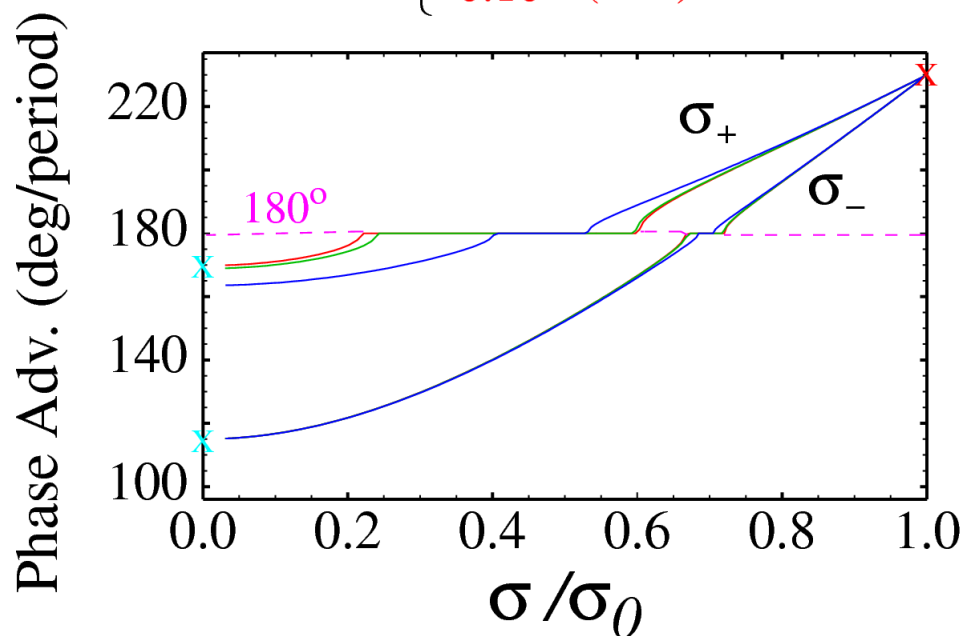
b) $\eta = 0.25, \sigma_0 = 115^\circ$

+: Stable +: Lattice Resonance
-: Stable -: Stable



Solenoidal Focusing – mode instability bands become wider and stronger for smaller occupancy

$$\eta = \begin{cases} 0.75 & \text{(Blue)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases} \quad \sigma_0 = 115^\circ$$



Comments:

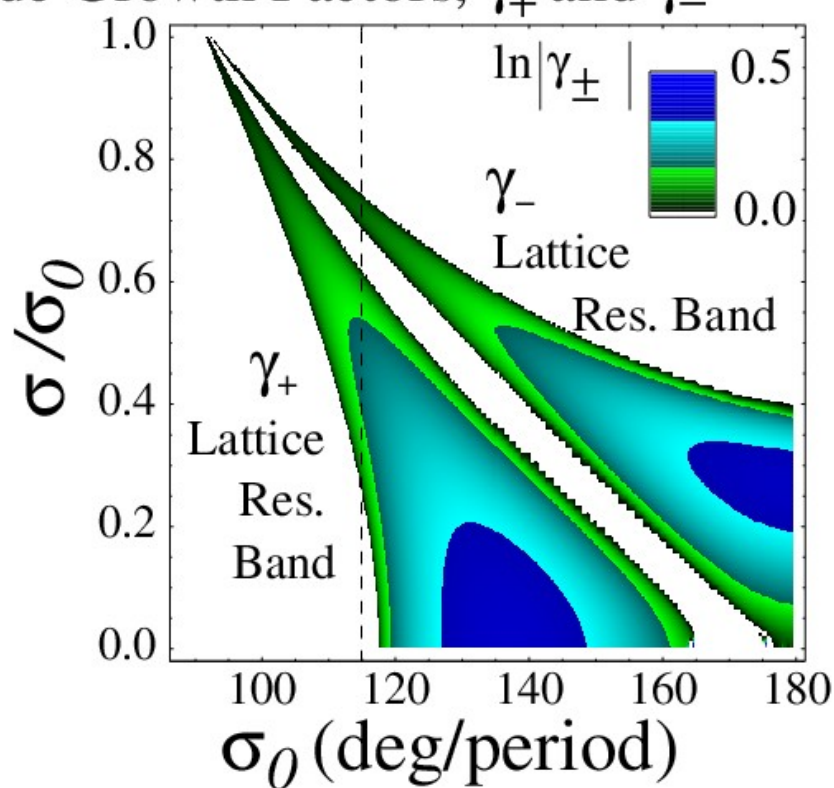
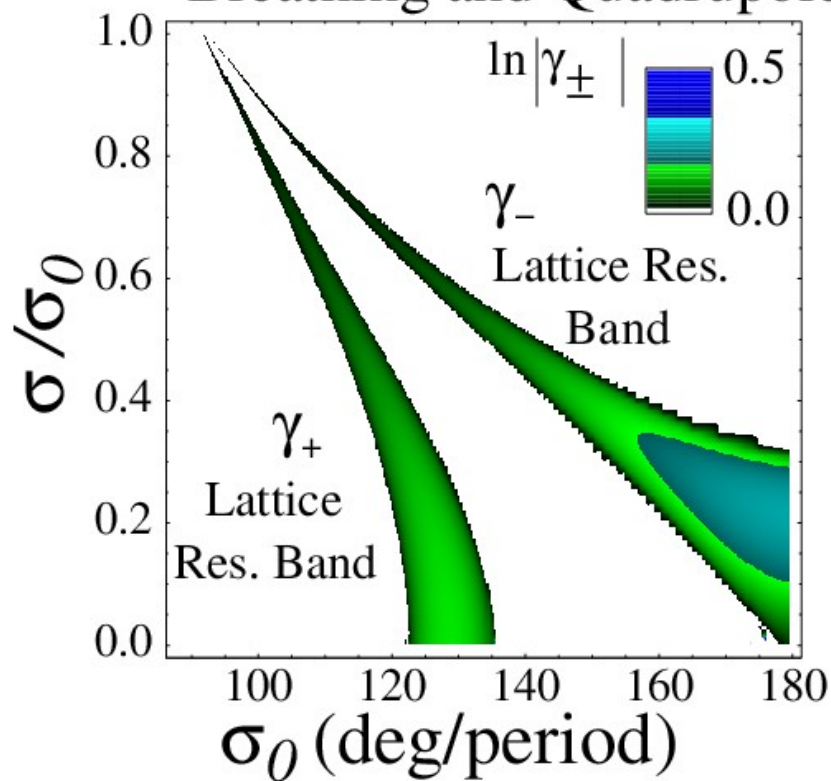
- ◆ Phase advance in instability band 180 deg.
- ◆ Significant deviations from continuous model even outside the band of instability when space-charge strong
- ◆ Instability band becomes stronger for low occupancy

Solenoidal Focusing – broad ranges of parametric instability are found for the breathing and quadrupole bands that must be avoided in machine operation

$$\eta = 0.75$$

$$\eta = 0.25$$

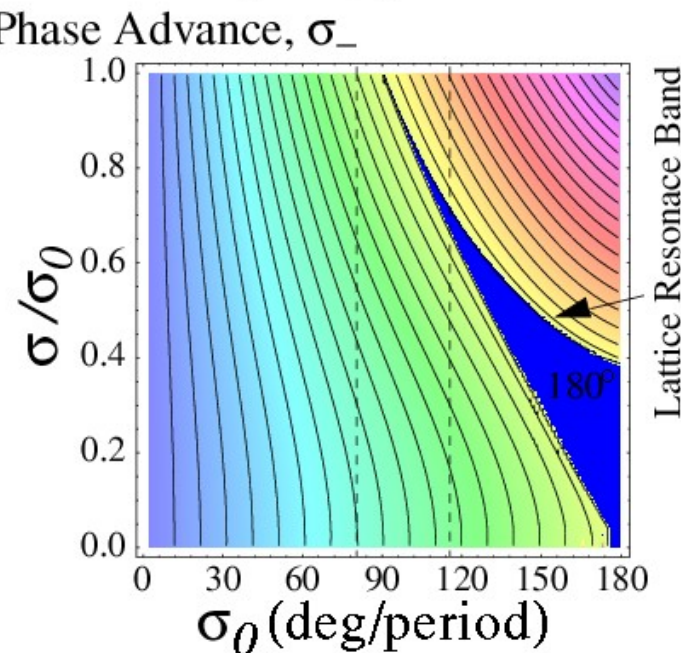
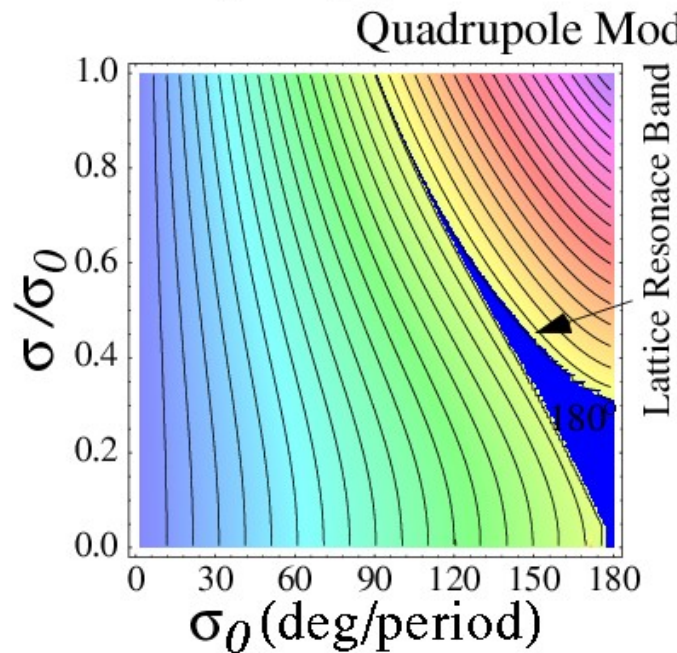
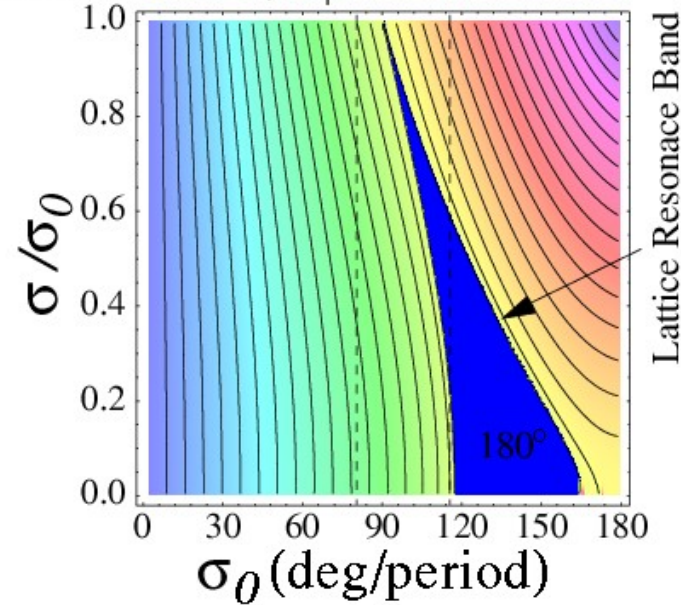
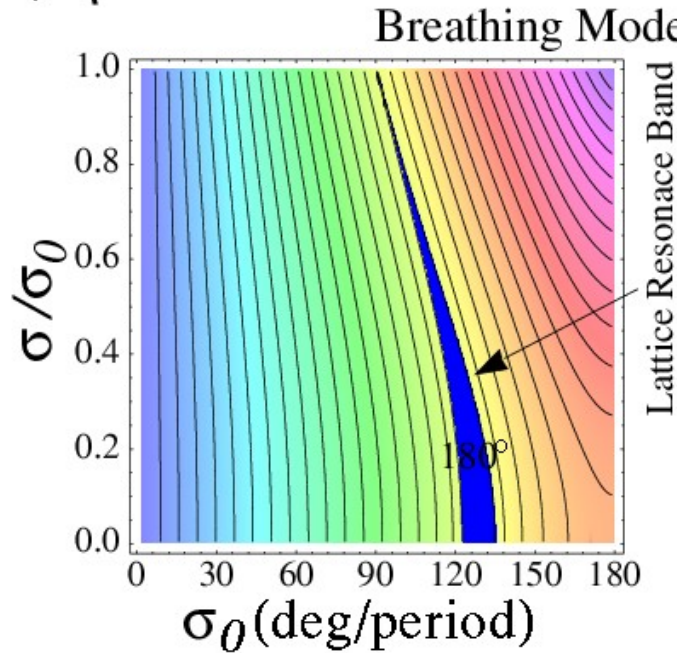
Breathing and Quadrupole Mode Growth Factors, γ_+ and γ_-



Solenoidal Focusing – parametric mode properties of band oscillations

a) $\eta = 0.75$

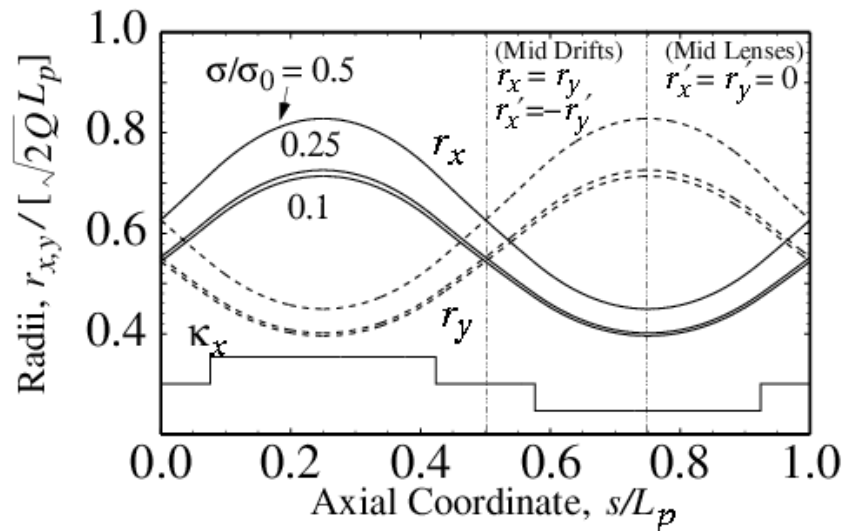
b) $\eta = 0.25$



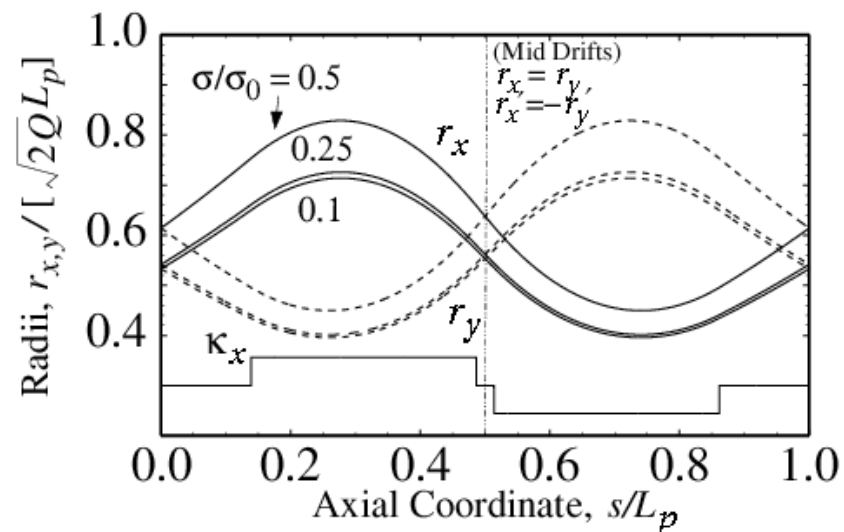
Quadrupole Doublet Focusing – Matched Envelope Solution

FODO and Syncopated Lattices

a) $\sigma_0 = 80^\circ$, $\eta = 0.6949$, and $\alpha = 1/2$ **FODO**



b) $\sigma_0 = 80^\circ$, $\eta = 0.6949$, and $\alpha = 0.1$ **Syncopated**



Focusing:

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

$$\kappa(s + L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

$$r_{xm}(s + L_p) = r_{xm}(s)$$

$$r_{ym}(s + L_p) = r_{ym}(s)$$

Comments:

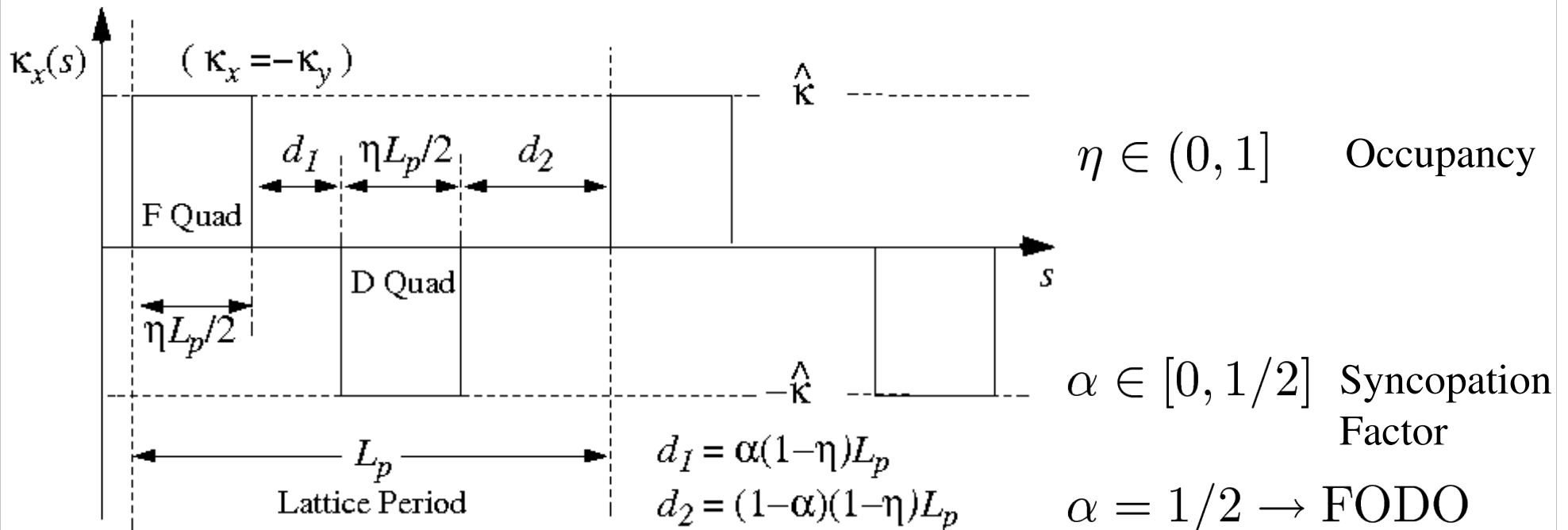
- ◆ Envelope flutter a *weak* function of occupancy η
- ◆ Syncopation factors $\alpha \neq 1/2$ reduce envelope symmetry and can drive more instabilities
- ◆ Space-charge expands envelope

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

See: S.M. Lund, lectures on **Transverse Particle Equations**

Quadrupole Doublet Focusing - piecewise constant focusing lattice

$$\cos \sigma_0 = \cos \Theta \cosh \Theta + \frac{1 - \eta}{\eta} \theta (\cos \Theta \sinh \Theta - \sin \Theta \cosh \Theta) - 2\alpha(1 - \alpha) \frac{(1 - \eta)^2}{\eta^2} \Theta^2 \sin \Theta \sinh \Theta \quad \Theta \equiv \frac{\sqrt{|\hat{\kappa}|} L_p}{2}$$

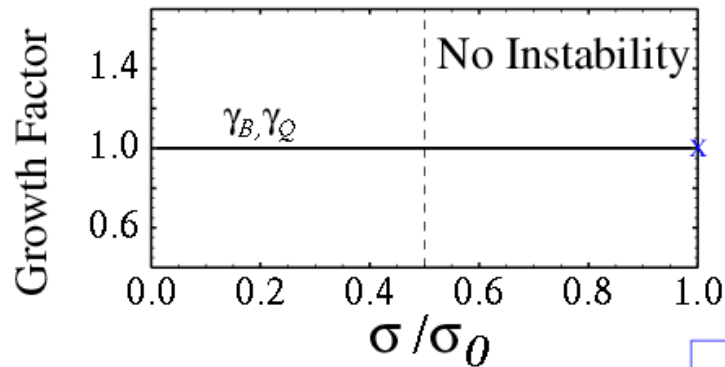
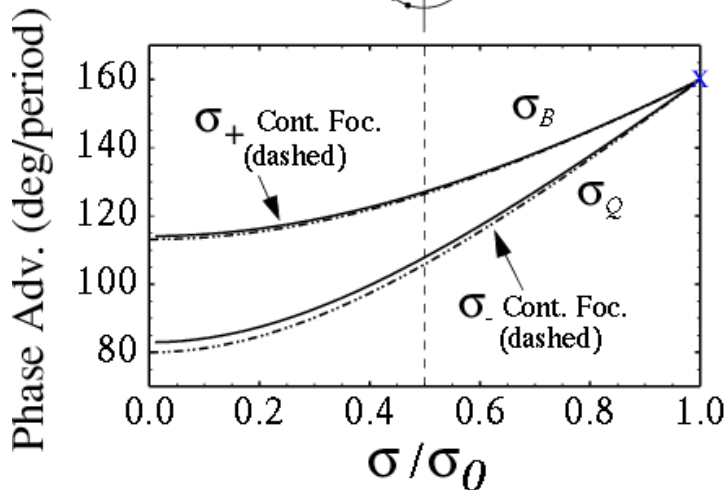
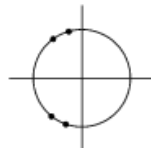


Quadrupole Focusing – parametric plots of breathing and quadrupole envelope mode phase advances two values of undepressed phase advance

a) $\eta = 0.6949$, $\alpha = 0.1$, $\sigma_0 = 80^\circ$

FODO

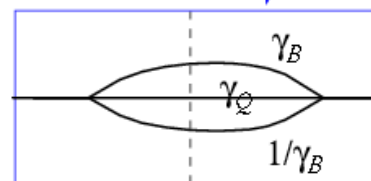
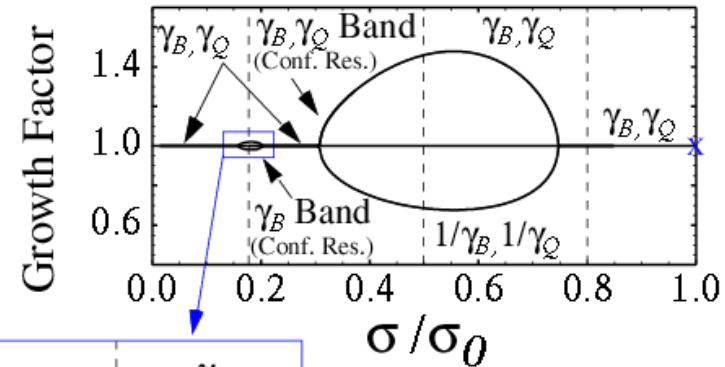
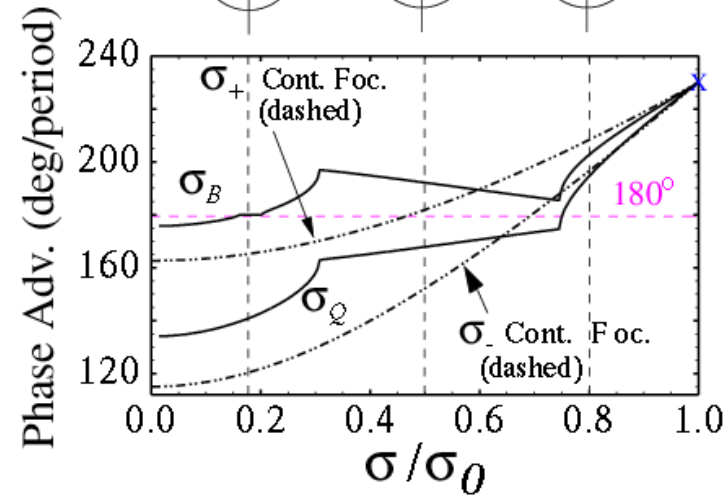
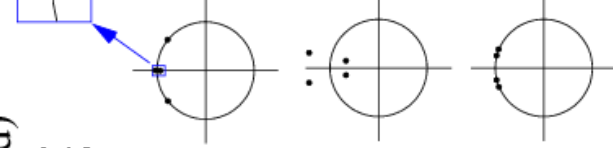
B: Stable
Q: Stable



b) $\eta = 0.6949$, $\alpha = 0.1$, $\sigma_0 = 115^\circ$

Syncopated

B: Lat. Res. B: Conf. Res. B: Stable
Q: Stable Q: Conf. Res. Q: Stable



Quadrupole Focusing – mode instability bands vary little/strongly with occupancy for FODO/syncopated lattices

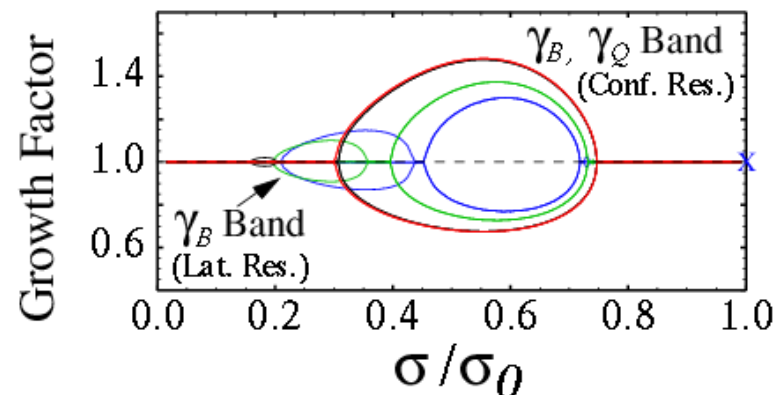
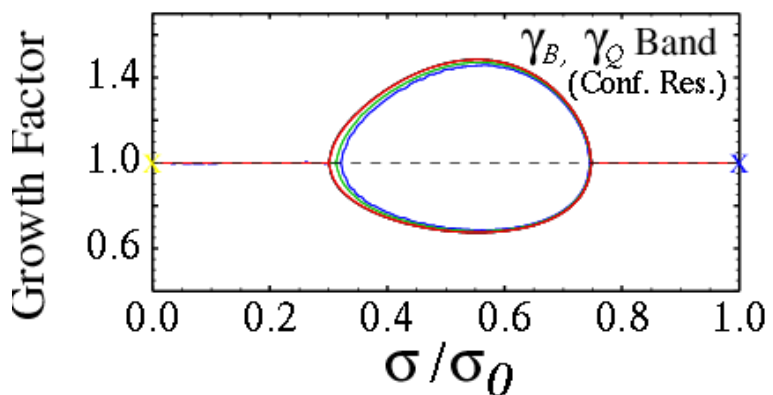
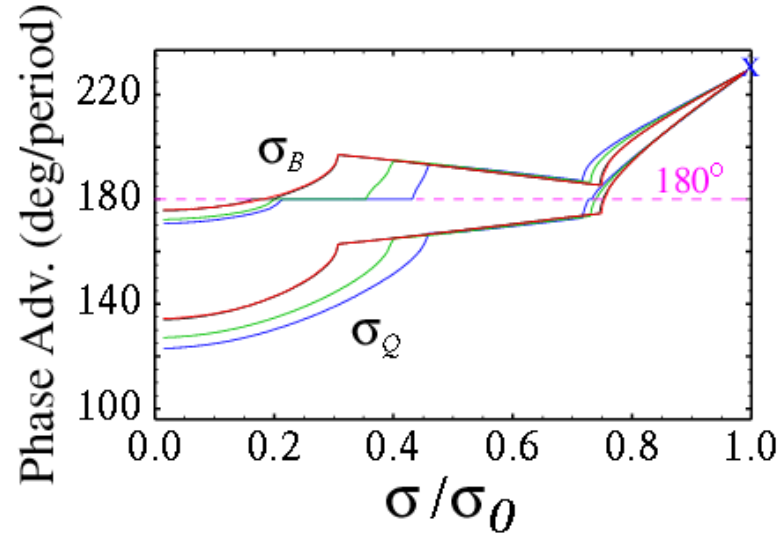
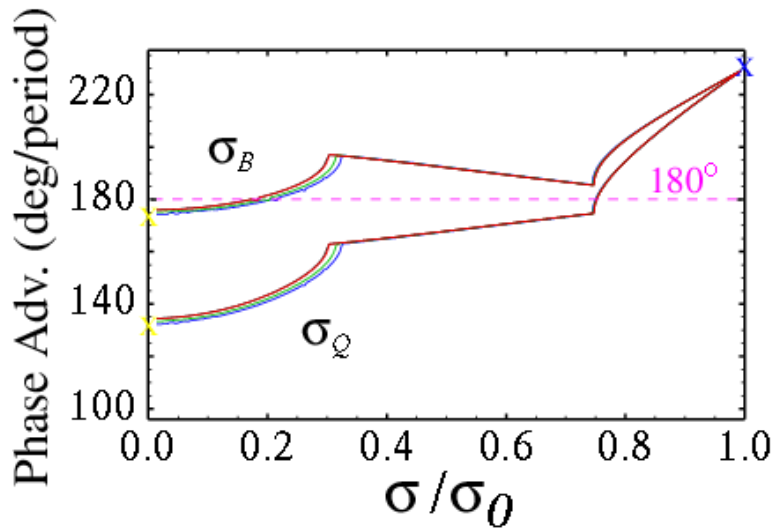
a) $\alpha = 1/2$ (FODO), $\sigma_0 = 115^\circ$

b) $\alpha = 0.1$, $\sigma_0 = 115^\circ$

FODO

Syncopated

$$\eta = \begin{cases} 0.90 & \text{(Blue)} \\ 0.6949 & \text{(Black)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases}$$



Quadrupole Focusing – broad ranges of parametric instability are found for the breathing and quadrupole bands that must be avoided in machine operation

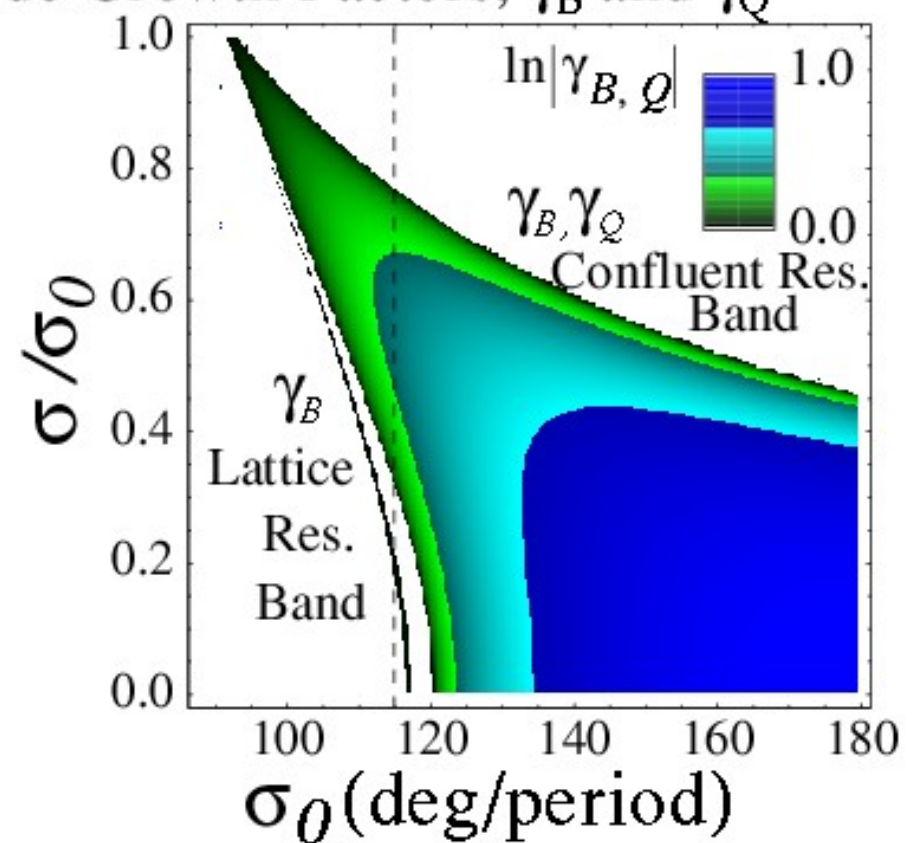
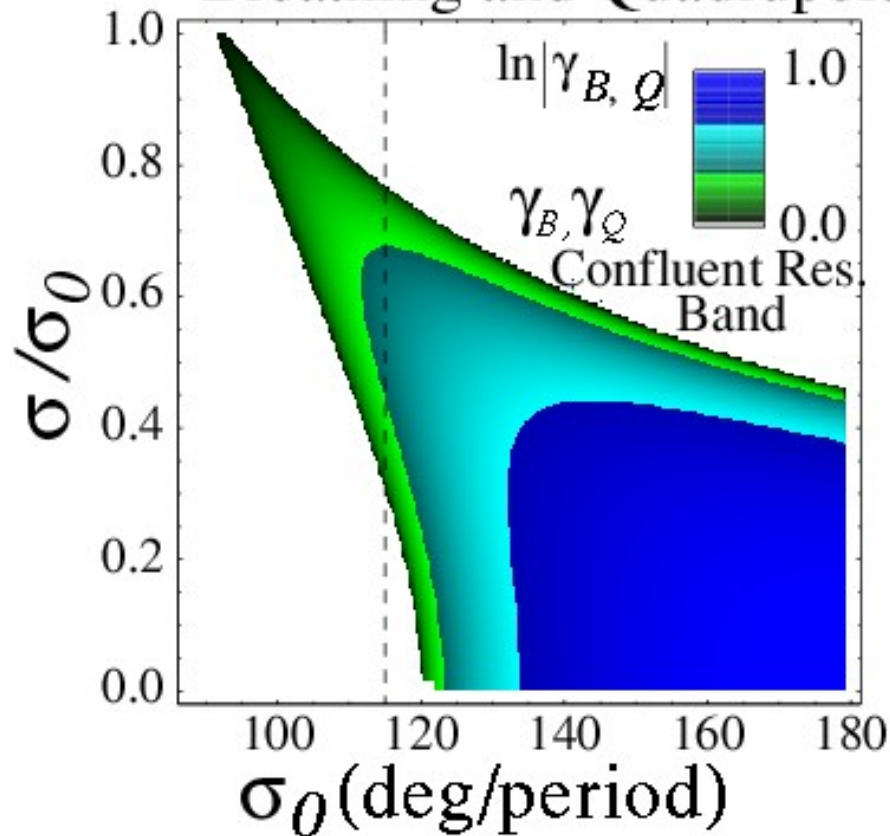
FODO Lattice

$$\eta = 0.6949, \quad \alpha = 1/2$$

Syncopated Lattice

$$\eta = 0.6949, \quad \alpha = 0.1$$

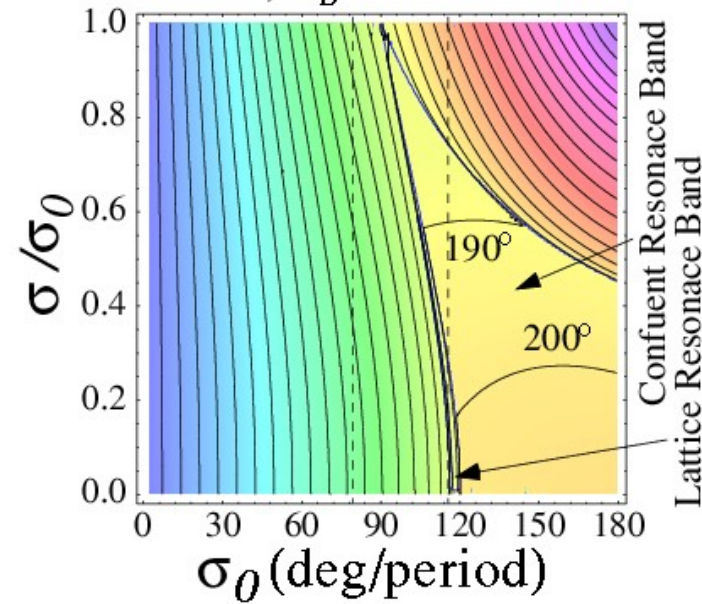
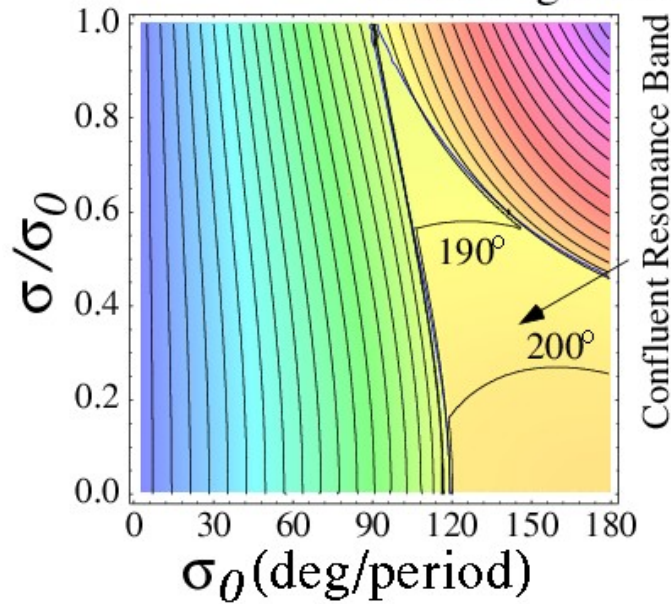
Breathing and Quadrupole Mode Growth Factors, γ_B and γ_Q



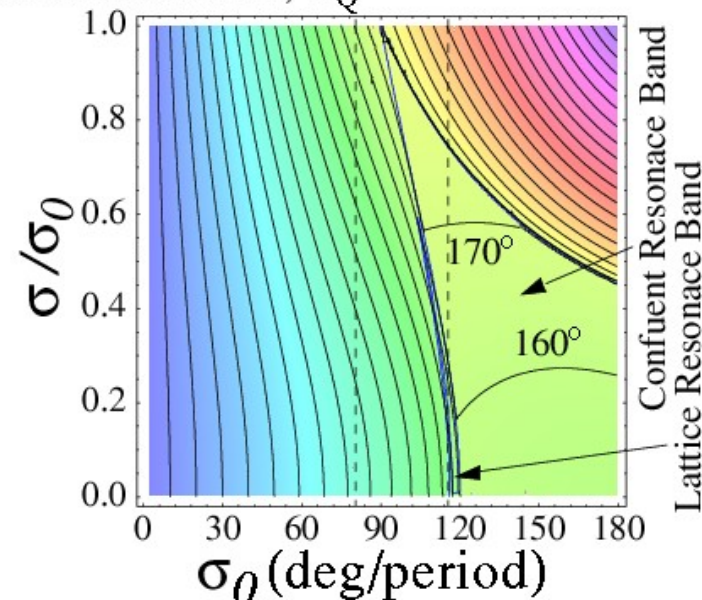
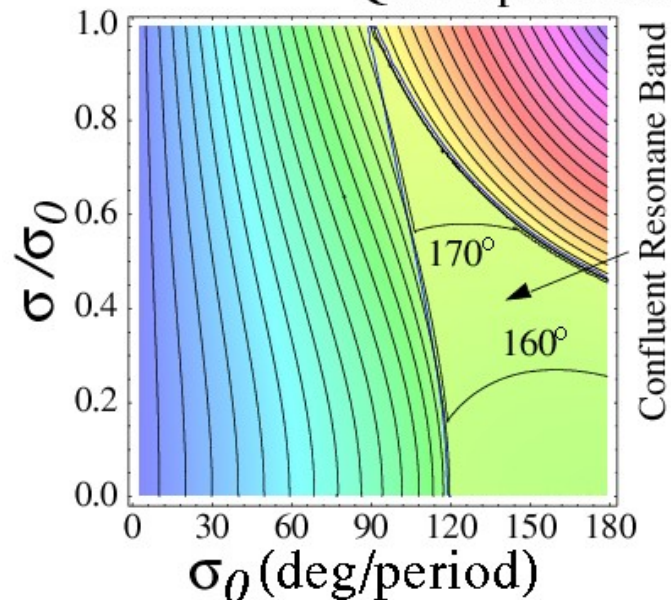
Quadrupole Focusing – parametric mode properties of band oscillations

a) $\eta = 0.6949$, $\alpha = 1/2$ **FODO** b) $\eta = 0.6949$, $\alpha = 0.1$ **Syncopated**

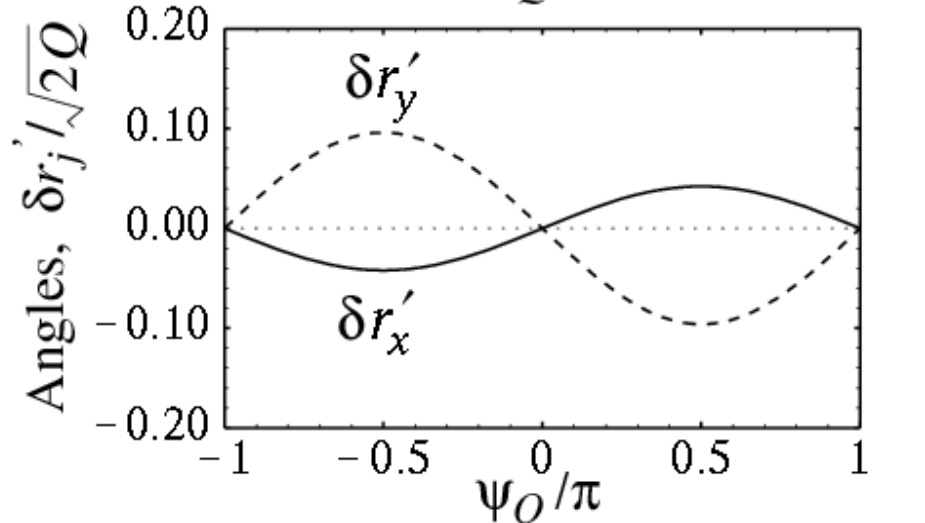
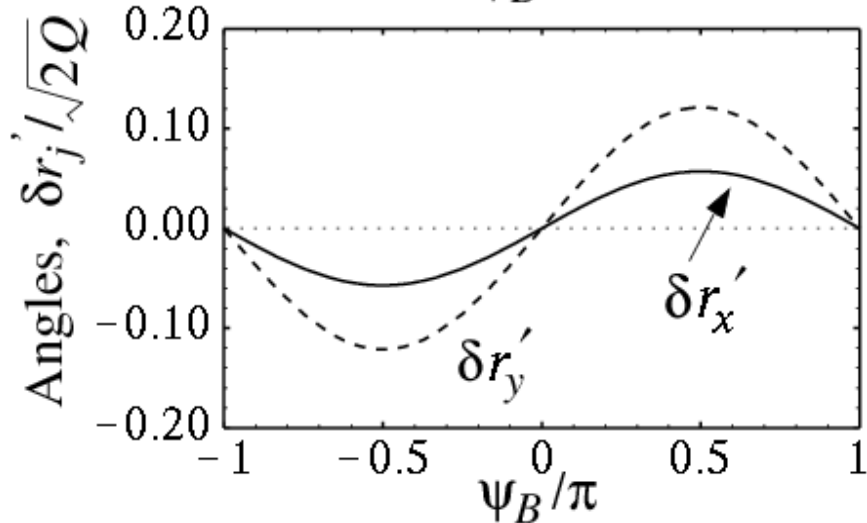
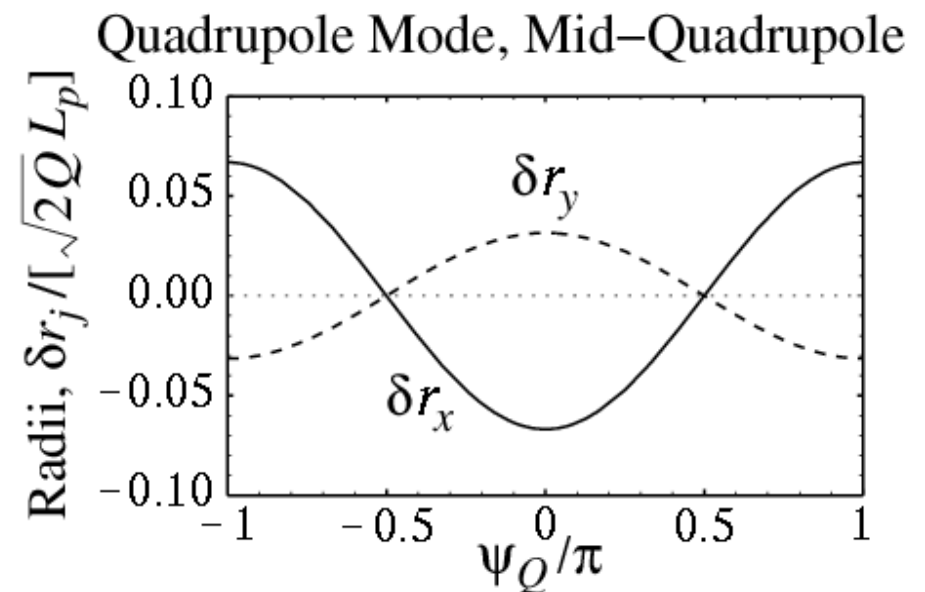
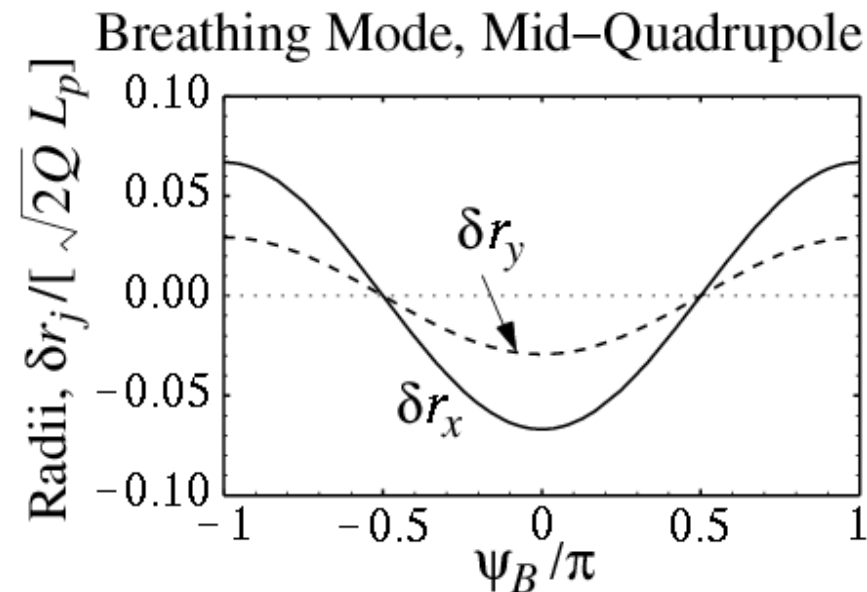
Breathing Mode Phase Advance, σ_B



Quadrupole Mode Phase Advance, σ_Q



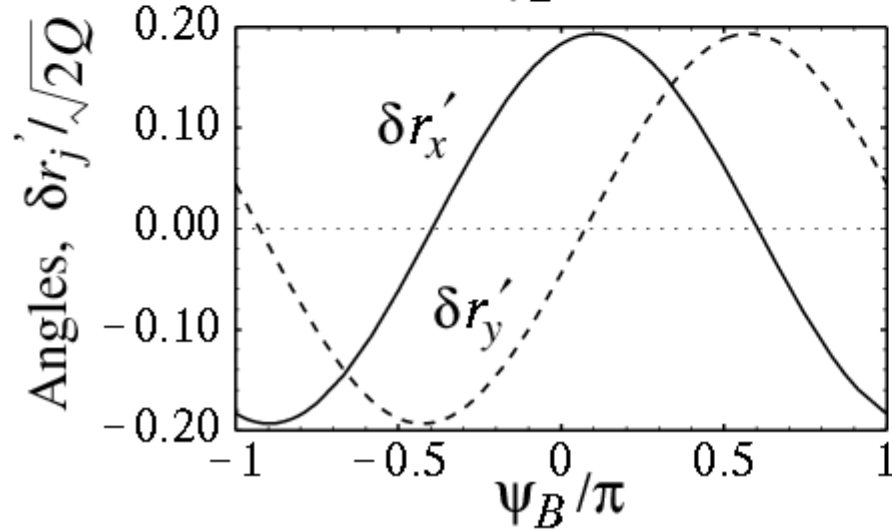
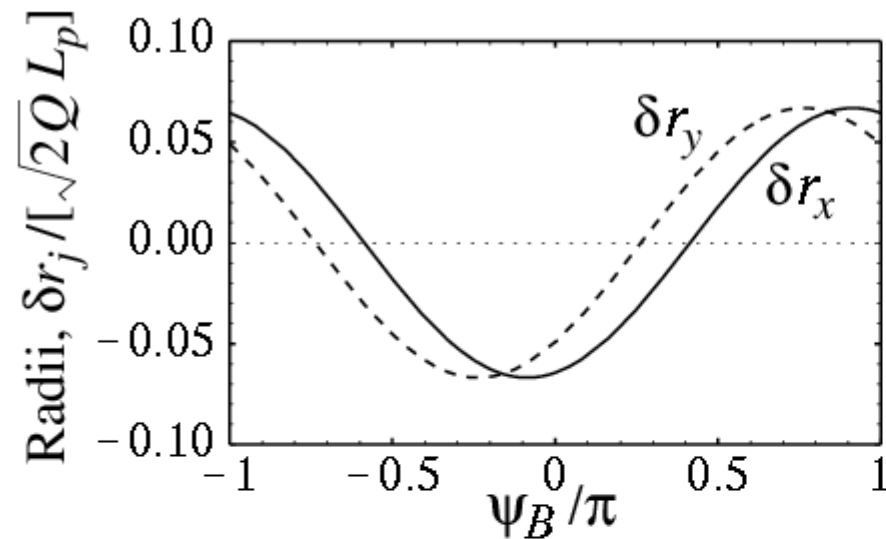
Quadrupole Focusing – mode structure varies strongly with mode phase and the location in the lattice (FODO example)



$\delta r_x \neq \delta r_y$
generally not exact
breathing symmetry

$\delta r_x \neq -\delta r_y$
generally not exact
quadrupole symmetry

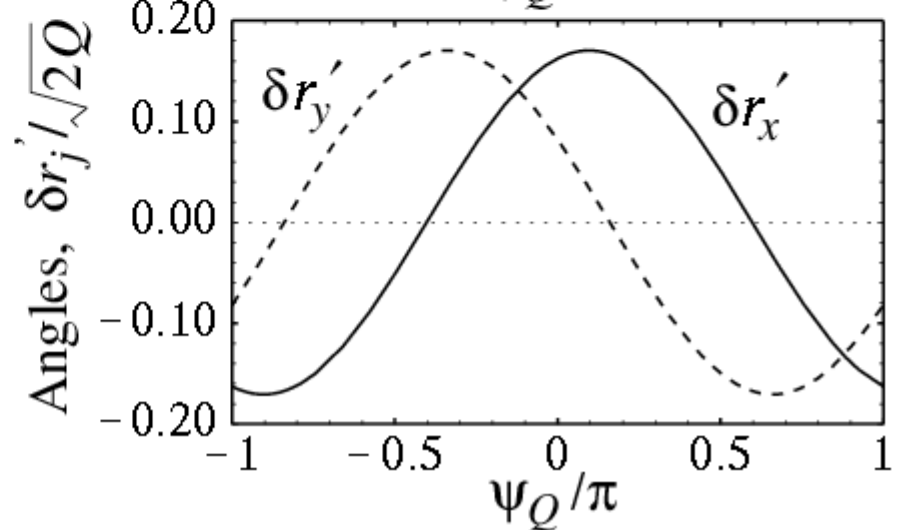
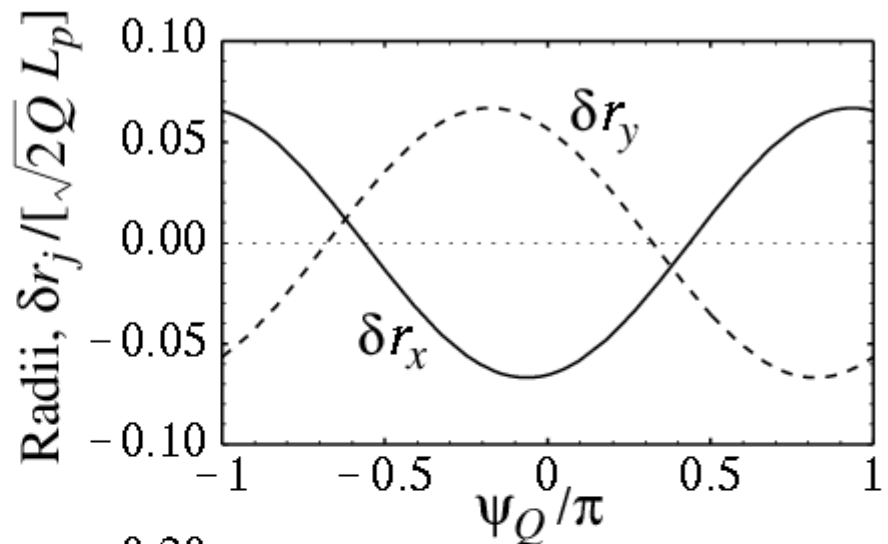
Breathing Mode, Mid-Drift



$$\delta r_x \neq \delta r_y$$

generally not exact
breathing symmetry

Quadrupole Mode, Mid-Drift



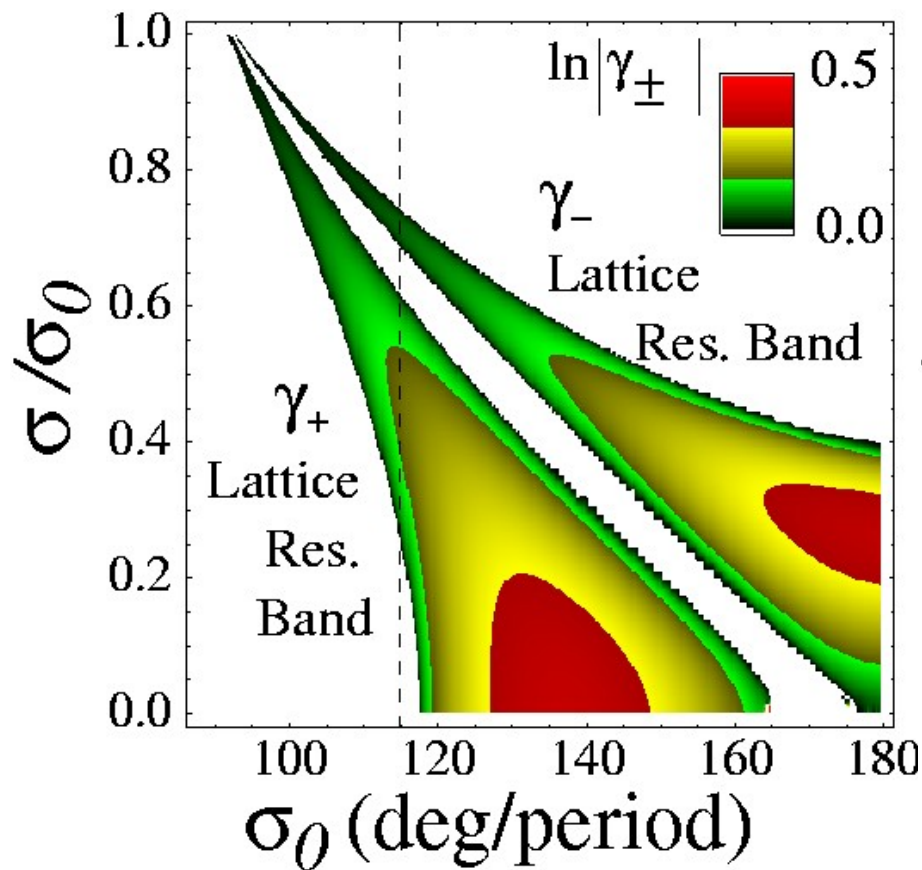
$$\delta r_x \neq -\delta r_y$$

generally not exact
quadrupole symmetry

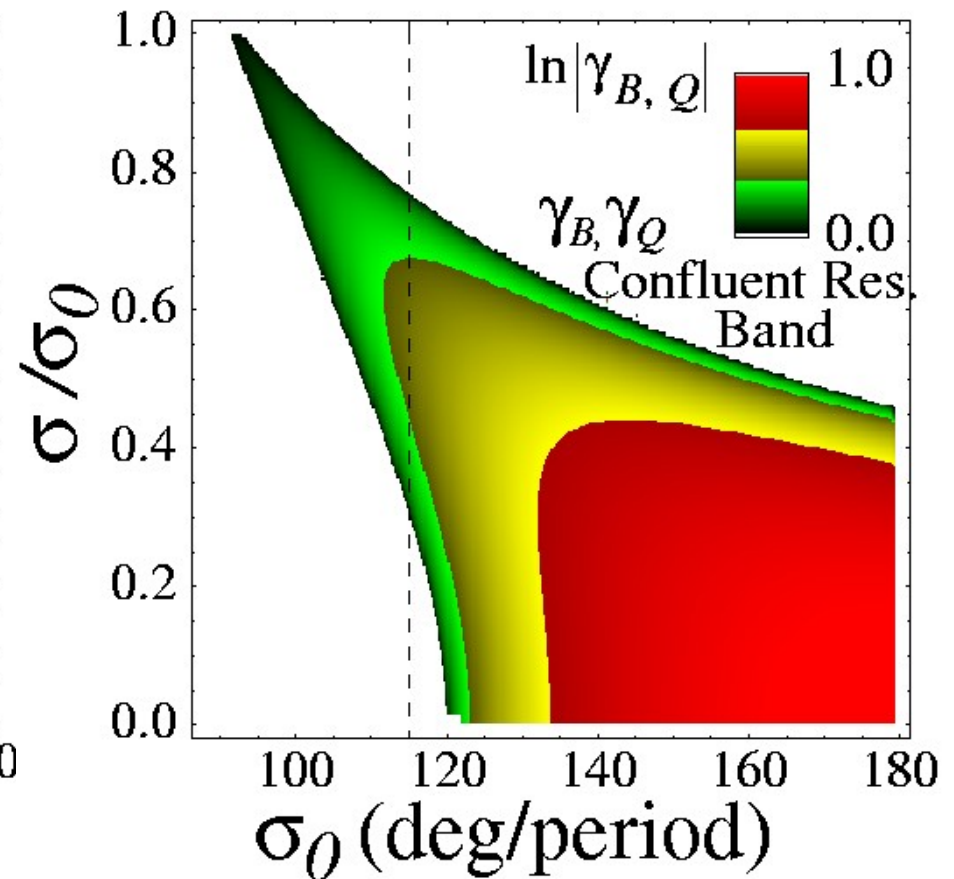
Summary: Envelope band instabilities and growth rates for periodic solenoidal and quadrupole doublet focusing lattices

Envelope Mode Instability Growth Rates

Solenoid ($\eta = 0.25$)



Quadrupole FODO ($\eta = 0.70$)



[S.M. Lund and B. Bukh, PRSTAB 024801 (2004)]

S9: Transport Limit Scaling Based on Envelope Models

See Handwritten Notes from 2008 USPAS

- ◆ Will attempt to convert to slides in future versions of the class

S8: Centroid and Envelope Descriptions via 1st Order Coupled Moment Equations

When constructing centroid and moment models, it can be efficient to simply write moments, differentiate them, and then apply the equation of motion. Generally, this results in lower order moments coupling to higher order ones and an infinite chain of equations. But the hierarchy can be truncated by:

- ◆ Assuming a fixed functional form of the distribution in terms of moments
- ◆ Neglecting coupling to higher order terms

Resulting first order moment equations can be expressed in terms of a closed set of moments and advanced in s or t using simple (ODE based) numerical codes. This approach can prove simpler to include effects where invariants are not easily extracted to reduce the form of the equations (as when solving the KV envelope equations in the usual form).

Examples of effects that might be more readily analyzed:

- ◆ Skew coupling in quadrupoles
- ◆ Chromatic effects in final focus
- ◆ Dispersion in bends

See: references at end of notes

J.J. Barnard, lecture on **Heavy-Ion
Fusion and Final Focusing**

Resulting form of coupled moment equations:

$$\frac{d}{ds} \mathbf{M} = \mathbf{F}(\mathbf{M})$$

\mathbf{M} = vector of moments, generally infinite

\mathbf{F} = vector function of \mathbf{M} , generally nonlinear

- ◆ System advanced from a specified initial condition (initial value of \mathbf{M})

Transverse moment definition:

$$\langle \dots \rangle_{\perp} \equiv \frac{\int d^2 x_{\perp} \int d^2 x'_{\perp} \dots f_{\perp}}{\int d^2 x_{\perp} \int d^2 x'_{\perp} f_{\perp}}$$

Can be generalized if other variables such as off momentum are included in f

Differentiate moments and apply equations of motion:

$$\frac{d}{ds} \langle \dots \rangle_{\perp} \equiv \frac{\int d^2 x_{\perp} \int d^2 x'_{\perp} \left[\frac{d}{ds} \dots \right] f_{\perp}}{\int d^2 x_{\perp} \int d^2 x'_{\perp} f_{\perp}}$$

+ apply equations of motion to simplify $\frac{d}{ds} \dots$

When simplifying the results, if the distribution form is frozen in terms of moments (Example: assume uniform density elliptical beam) then we use constructs like:

$$n = \int d^2 x'_\perp f_\perp = n(\mathbf{M})$$

to simplify the resulting equations and express the RHS in terms of elements of \mathbf{M}

1st order moments:

$$\mathbf{X}_\perp = \langle \mathbf{x}_\perp \rangle_\perp \quad \text{Centroid coordinate}$$

$$\mathbf{X}'_\perp = \langle \mathbf{x}'_\perp \rangle_\perp \quad \text{Centroid angle}$$

+ possible others if more variables. Example

$$\Delta = \left\langle \frac{\delta p_s}{p_s} \right\rangle = \langle \delta \rangle \quad \text{Centroid off-momentum}$$

$$\vdots \quad \quad \quad \vdots$$

2nd order moments:

It is typically convenient to subtract centroid from higher-order moments

$$\tilde{x} \equiv x - X \quad \tilde{x}' \equiv x' - X'$$

$$\tilde{y} \equiv y - Y \quad \tilde{y}' \equiv y' - Y'$$

$$\tilde{\delta} \equiv \delta - \Delta$$

x-moments y-moments x-y cross moments dispersive moments

$$\begin{array}{cccc} \langle \tilde{x}^2 \rangle_{\perp} & \langle \tilde{y}^2 \rangle_{\perp} & \langle \tilde{x}\tilde{y} \rangle_{\perp} & \langle \tilde{x}\tilde{\delta} \rangle, \langle \tilde{y}\tilde{\delta} \rangle \\ \langle \tilde{x}\tilde{x}' \rangle_{\perp} & \langle \tilde{y}\tilde{y}' \rangle_{\perp} & \langle \tilde{x}'\tilde{y} \rangle_{\perp}, \langle \tilde{x}\tilde{y}' \rangle_{\perp} & \langle \tilde{x}'\tilde{\delta} \rangle, \langle \tilde{y}'\tilde{\delta} \rangle \\ \langle \tilde{x}'^2 \rangle_{\perp} & \langle \tilde{y}'^2 \rangle_{\perp} & \langle \tilde{x}'\tilde{y}' \rangle_{\perp} & \langle \tilde{\delta}^2 \rangle \end{array}$$

3rd order moments: Analogous to 2nd order case, but more for each order

$$\langle \tilde{x}^3 \rangle_{\perp}, \langle \tilde{x}^2\tilde{y} \rangle_{\perp}, \dots$$

Many quantities of physical interest are expressed in transport can then be expressed in terms of moments calculated when the equations are numerically advanced in s and their evolutions plotted to understand behavior

- ◆ Many quantities of physical interest are expressible in terms of 1st and 2nd order moments

Example moments often projected:

Statistical beam size:
(rms edge measure)

$$r_x = 2 \langle \tilde{x}^2 \rangle_{\perp}^{1/2}$$
$$r_y = 2 \langle \tilde{y}^2 \rangle_{\perp}^{1/2}$$

Statistical emittances:
(rms edge measure)

$$\varepsilon_x = 4 \left[\langle \tilde{x}^2 \rangle_{\perp} \langle \tilde{x}'^2 \rangle_{\perp} - \langle \tilde{x} \tilde{x}' \rangle_{\perp}^2 \right]^{1/2}$$
$$\varepsilon_y = 4 \left[\langle \tilde{y}^2 \rangle_{\perp} \langle \tilde{y}'^2 \rangle_{\perp} - \langle \tilde{y} \tilde{y}' \rangle_{\perp}^2 \right]^{1/2}$$

Kinetic longitudinal temperature:
(rms measure)

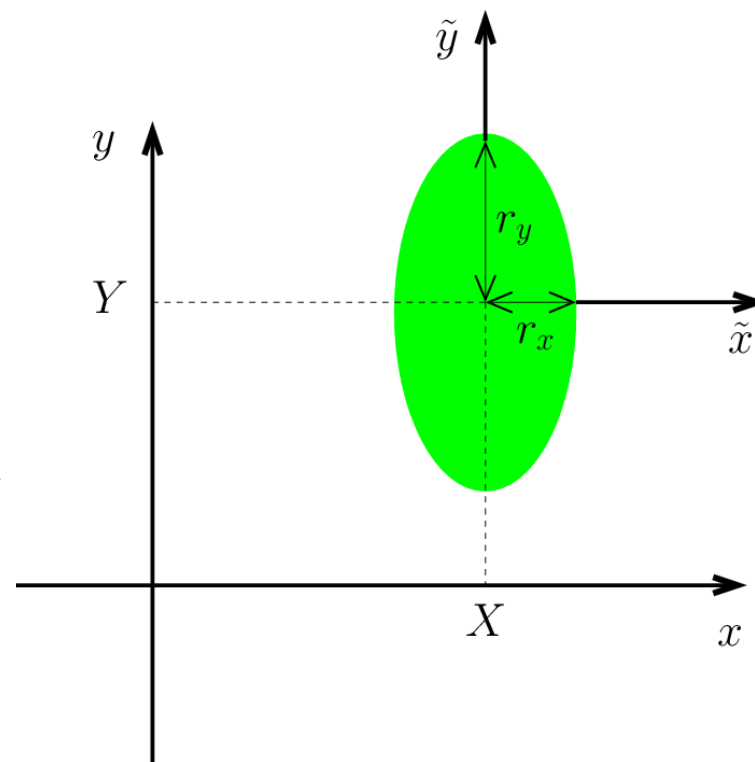
$$T_s = \text{const} \times \langle \tilde{\delta}^2 \rangle$$

Illustrate approach with the familiar KV model

Truncation assumption: unbunched uniform density elliptical beam in free space

- ◆ $\delta = 0$, no axial velocity spread
- ◆ All cross moments zero, i.e. $\langle \tilde{x}\tilde{y} \rangle_{\perp} = 0$

$$\begin{aligned} \frac{d}{ds} \langle x \rangle_{\perp} &= \langle x' \rangle_{\perp} & \frac{d}{ds} \langle x^2 \rangle_{\perp} &= 2 \langle x x' \rangle_{\perp} \\ \frac{d}{ds} \langle x' \rangle_{\perp} &= \langle x'' \rangle_{\perp} & \frac{d}{ds} \langle x'^2 \rangle_{\perp} &= 2 \langle x' x'' \rangle_{\perp} \\ &\vdots & &\vdots \end{aligned}$$



Use particle equations of motion within beam, neglect images, and simplify

- ◆ Apply equations in **S2** with $\mathbf{E}_{\perp}^i = 0$

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + \kappa_x x - \frac{2Q}{(r_x + r_y)r_x} (x - \langle x \rangle_{\perp}) = 0$$

$$y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + \kappa_y y - \frac{2Q}{(r_x + r_y)r_y} (y - \langle y \rangle_{\perp}) = 0$$

Resulting system of 1st and 2nd order moments

1st order moments:

$$\frac{d}{ds} \begin{bmatrix} \langle x \rangle_{\perp} \\ \langle x' \rangle_{\perp} \\ \langle y \rangle_{\perp} \\ \langle y' \rangle_{\perp} \end{bmatrix} = \begin{bmatrix} \langle x' \rangle_{\perp} \\ -\kappa_x(s) \langle x \rangle_{\perp} \\ \langle y' \rangle_{\perp} \\ -\kappa_y(s) \langle y \rangle_{\perp} \end{bmatrix}$$

2nd order moments:

$$\frac{d}{ds} \begin{bmatrix} \langle \tilde{x}^2 \rangle_{\perp} \\ \langle \tilde{x}\tilde{x}' \rangle_{\perp} \\ \langle \tilde{x}'^2 \rangle_{\perp} \\ \langle \tilde{y}^2 \rangle_{\perp} \\ \langle \tilde{y}\tilde{y}' \rangle_{\perp} \\ \langle \tilde{y}'^2 \rangle_{\perp} \end{bmatrix} = \begin{bmatrix} 2 \langle \tilde{x}\tilde{x}' \rangle_{\perp} \\ \langle \tilde{x}'^2 \rangle_{\perp} - \kappa_x(s) \langle \tilde{x}^2 \rangle_{\perp} + \frac{Q \langle \tilde{x}'^2 \rangle_{\perp}}{[4 \langle \tilde{x}^2 \rangle_{\perp}^{1/2} (\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ -2\kappa_x(s) \langle \tilde{x}\tilde{x}' \rangle_{\perp} + \frac{2Q \langle \tilde{x}\tilde{x}' \rangle_{\perp}}{[4 \langle \tilde{x}^2 \rangle_{\perp}^{1/2} (\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ 2 \langle \tilde{y}\tilde{y}' \rangle_{\perp} \\ \langle \tilde{y}'^2 \rangle_{\perp} - \kappa_y(s) \langle \tilde{y}^2 \rangle_{\perp} + \frac{Q \langle \tilde{y}'^2 \rangle_{\perp}}{[4 \langle \tilde{y}^2 \rangle_{\perp}^{1/2} (\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ -2\kappa_y(s) \langle \tilde{y}\tilde{y}' \rangle_{\perp} + \frac{2Q \langle \tilde{y}\tilde{y}' \rangle_{\perp}}{[4 \langle \tilde{y}^2 \rangle_{\perp}^{1/2} (\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \end{bmatrix}$$

- ◆ Express 1st and 2nd order moments separately in this case since uncoupled
- ◆ Form truncates due to frozen distribution form: all moments on LHS on RHS
- ◆ Integrate from initial moments values of s and project out desired quantities

Using 2nd order moment equations we can show that

$$\frac{d}{ds}\varepsilon_x^2 = 0 = \frac{d}{ds}\varepsilon_y^2$$

$$\varepsilon_x^2 = 16 \left[\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2 \right] = \text{const}$$

$$\implies \varepsilon_y^2 = 16 \left[\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}^2 \right] = \text{const}$$

Using this, the 2nd order moment equations can be equivalently expressed in the standard KV envelope form:

$$\frac{dr_x}{ds} = r'_x ; \quad \frac{d}{ds}r'_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$\frac{dr_y}{ds} = r'_y ; \quad \frac{d}{ds}r'_y + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

- ◆ Moment form fully consistent with usual KV model ... as it must be
- ◆ Moment form generally easier to put in additional effects that would violate the usual emittance invariants

Relative advantages of the use of coupled matrix form versus reduced equations can depend on the problem/situation

Coupled Matrix Equations

$$\frac{d}{ds}\mathbf{M} = \mathbf{F}$$

\mathbf{M} = Moment Vector

\mathbf{F} = Force Vector

- ◆ Easy to formulate
 - Straightforward to incorporate additional effects
- ◆ Natural fit to numerical routine
 - Easy to code

Reduced Equations

$$X'' + \kappa_x X = 0$$

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

etc.

Reduction based on identifying invariants such as

$$\varepsilon_x^2 = 16 \left[\langle \tilde{x}^2 \rangle_{\perp} \langle \tilde{x}'^2 \rangle_{\perp} - \langle \tilde{x}\tilde{x}' \rangle_{\perp}^2 \right]$$

helps understand solutions

- ◆ Compact expressions

These notes will be corrected and expanded for reference and future editions of US Particle Accelerator School and University of California at Berkeley courses:

“Beam Physics with Intense Space Charge”

*“Interaction of Intense Charged Particle Beams
with Electric and Magnetic Fields”*

by J.J. Barnard and S.M. Lund

Corrections and suggestions for improvements are welcome. Contact:

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(510) 486 – 6936

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E.P. Lee, E. Close, and L. Smith, *Nuc. Inst. And Methods*, 1126 (1987)

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S.M. Lund and S.H. Chilton, and E.P. Lee, *Efficient Computation of Matched Solutions of the KV Envelope Equations*, PRSTAB **9** 064201 (2006)

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A. Dragt, *Lectures on Nonlinear Orbit Dynamics in Physics of High Energy Particle Accelerators*, (American Institute of Physics, 1982), AIP Conf. Proc. No. 87, p. 147

E. D. Courant and H. S. Snyder, *Theory of the Alternating-Gradient Synchrotron*, *Annals of Physics* **3**, 1 (1958)

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