Problem Set #6
NE 290 H Barnard and Lund
Due March 9, 2009

S. M. Lund P6/

Problem #1
15 pts

TED Problem 6

6/ For continuous focusing equilibria it was shown that:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0 \gamma_b^3 \beta_b^2 c^2}\int_{\psi(r)}^{\infty} dH_\perp f_\perp(H_\perp)$$

$$\psi(r=0) = 0$$

a) Apply this formula to the thermal equilibrium distribution

$$f_\perp = \frac{\gamma_b m \beta_b^2 c^2 \hat{n}}{2\pi T}\exp\left\{-\frac{\gamma_b m \beta_b^2 c^2 H_\perp}{T}\right\}$$

to derive the transformed thermal equilibrium Poisson equation presented in class:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial \tilde{\psi}}{\partial \rho}\right] = 1 + \Delta - e^{-\tilde{\psi}}$$

b) Show that the thermal equilibrium distribution satisfies the Density Inversion Theorem:

$$f_\perp(H_\perp) = -\frac{1}{2\pi}\left.\frac{\partial n}{\partial \psi}\right|_{\psi = H_\perp}$$

c) Verify the thermal equilibrium formula:

$$\varepsilon_x^2 = 16\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right] = \frac{16T}{\gamma_b m \beta_b^2 c^2}\langle x^2\rangle_\perp$$

$$= \frac{4T}{\gamma_b m \beta_b^2 c^2} r_b^2$$

Hint for $\alpha > 0$:

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

Take $\partial/\partial\alpha$ for other needed formulas.

TED Problem 7

Problem #2
15 pts

S.M. Lund

7/ For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$$

and a round, "matched" KV equilibrium beam with

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\,\delta(H_\perp - H_b)$$

where we have:

$$H_\perp = \frac{1}{2}(x'^2 + y'^2) + \frac{k_{\beta 0}^2}{2}(x^2+y^2) + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$= \frac{1}{2}(x'^2+y'^2) + \frac{\varepsilon_x^2}{2r_b^4}(x^2+y^2)$$

and

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

$$H_b = \frac{\varepsilon_x^2}{2r_b^2}$$

a) Calculate within the beam core $(0 \le r < r_b)$ the moment:

$$\langle x'^2 \rangle_{\vec{x}_\perp'} \equiv \frac{\int d^2x'\, x'^2 f_\perp(H_\perp)}{\int d^2x'\, f_\perp(H_\perp)}$$

You can use results from previous problems.

b) What is the value of this moment at the beam edge $(r = r_b)$? Is this value consistent with what one expects for a sharp beam edge? Why?

TPR Problem 1 | Problem #3 20 pts | S.M. Lund | P.V

1/ Consider the driven harmonic oscillator equation for $U(\varphi)$:

$$\frac{d^2U(\varphi)}{d\varphi^2} + \nu_0^2 U(\varphi) = \overbrace{A\cos(\nu\varphi) + B\sin(\nu\varphi)}^{\text{driving term.}}$$

ν = constant driving frequency.
A, B constant amplitudes.

The general solution for $U(\varphi)$ can be expanded as

$$U(\varphi) = U_h(\varphi) + U_p(\varphi)$$

where U_h is the general solution to the homogeneous equation:

$$\frac{d^2U_h}{d\varphi^2} + \nu_0^2 U_h = 0$$

$$\Rightarrow U_h = C_1\cos(\nu_0\varphi) + C_2\sin(\nu_0\varphi)$$

C_1, C_2 constants

and U_p is any particular solution to

$$\frac{d^2U_p}{d\varphi^2} + \nu_0^2 U_p = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

a) For $\nu \neq \nu_0$ show that a solution U_p exists proportional to the driving term and find the constant of proportionality.

b) Use the results of part a) to construct the solution ($\nu \neq \nu_0$) for $U(\varphi)$ satisfying the initial conditions at $\varphi = 0$:

$$U(\varphi=0) = U_0$$

$$\left.\frac{dU}{d\varphi}\right|_{\varphi=0} = \dot{U}_0 \quad ; \quad \frac{dU}{d\varphi} \equiv \dot{U}$$

c) Set $\nu = \nu_0 + \delta\nu$ and find the leading order form of the solution valid for $|\delta\nu|/\nu_0 \ll 1$ and $|\delta\nu\,\varphi| \ll 1$.
What does this limit imply on the amplitude of the particle oscillation as $\nu \to \nu_0$

d) What do these results imply for a general periodic forcing function:

$$\frac{d^2 U(\varphi)}{d\varphi^2} + \nu_0^2 U(\varphi) = f(\varphi) \quad \leftarrow \text{forcing function}$$

$$f(\varphi + 2\pi) = f(\varphi)$$

How does this fit in with the analysis of machine tunes carried out in the class notes?

Problem # 4
10 pts.

TPR Problem 2

S.M. Lund

2

Consider a ring composed of N identical lattice periods and with:

L_p = lattice period length

σ_0 = phase adv of single particle in x or y dir.

a) What is the tune $\nu_0 = \nu_{0x} = \nu_{0y}$?

b) If we model the x-focusing as continuous, and assume a transverse matched KV beam with:

r_b = const matched beam radius

Q = const perveance.

then what is the depressed tune $\nu = \nu_x = \nu_y$?

Hints: 1) Use formulas in Transverse Equilibrium distributions for a matched KV cont. focusing beam to derive a formula for the depressed phase advance k_β in terms of $k_{\beta 0}$, Q, r_b

2) Take $k_{\beta 0}^2 = (\sigma_0/L_p)^2$ to model continuous focusing.

3) Write a formula for ν based on k_β using logic of part a)

c) If we allow Q to vary, what is the maximum value of Q that can be transported for fixed L_p, N, σ_0 and r_b based on the Laslett space-charge limit?