Non-Abelian Courant-Snyder Theory for Coupled Transverse Dynamics

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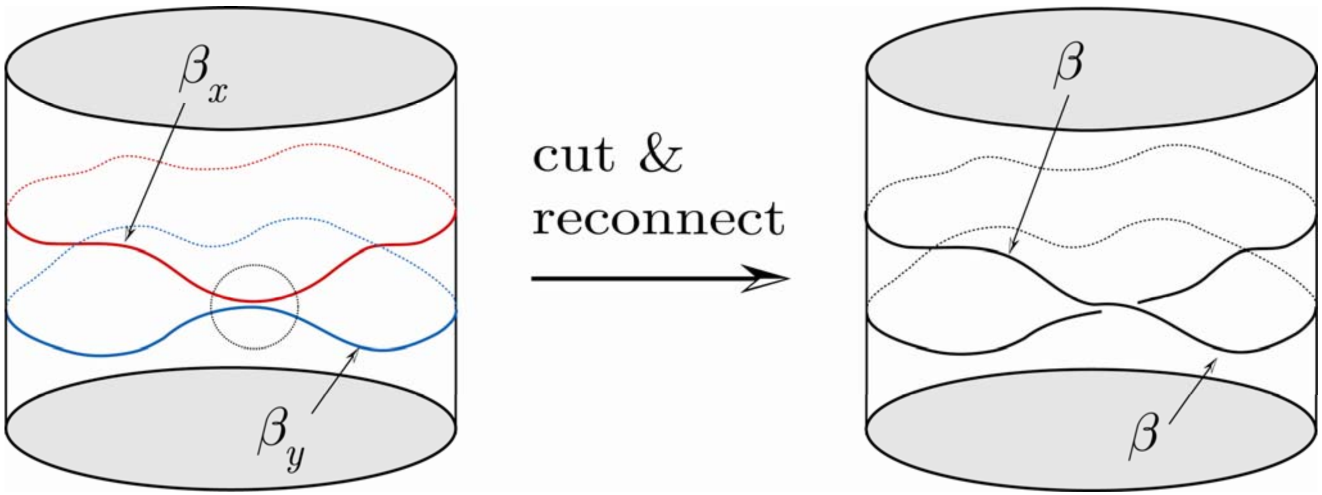
How to make a smooth round beam?

- Solenoid
  - Final focus (NDCX)
  - How to match quadrupole with solenoid (NDCX-III)?
- Skew-quadrupole

Möbius Accelerator

[Talman, PRL 95]



- Round beam, one tune, one chromaticity
- How?
  - Solenoid or skew-quadrupole
- What is going on during the flip?

Coupled transverse dynamics (2 degree of freedom)

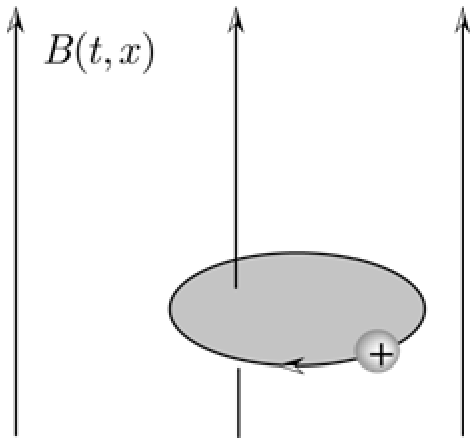
$$H_c = zA_c z^T\,, \quad z = (\,x, y, \dot{x}, \dot{y}\,)$$

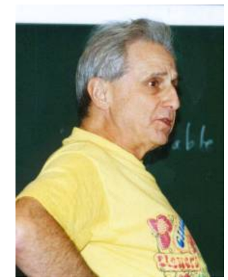
$$A_c = \begin{pmatrix} \kappa & R \\ R^T & \dfrac{I}{2} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}$$

solenoidal, quadrupole,
& skew-quadrupole

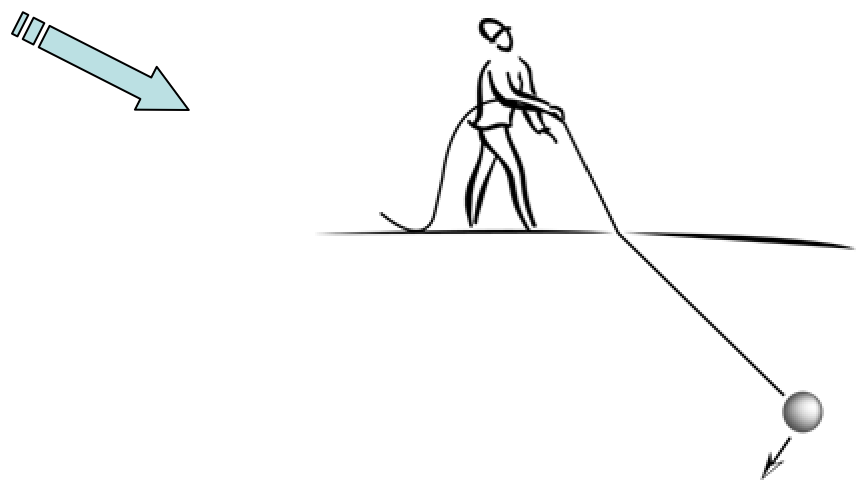
$$\kappa = \begin{pmatrix} \dfrac{\Omega^2}{2} + \dfrac{\kappa_q}{2} & \kappa_{sq} \\ \kappa_{sq} & \dfrac{\Omega^2}{2} - \dfrac{\kappa_q}{2} \end{pmatrix}, \; R = \begin{pmatrix} 0 & -\dfrac{\Omega}{2} \\ \dfrac{\Omega}{2} & 0 \end{pmatrix}$$

Similar 2D problem – adiabatic invariant of gyromotion



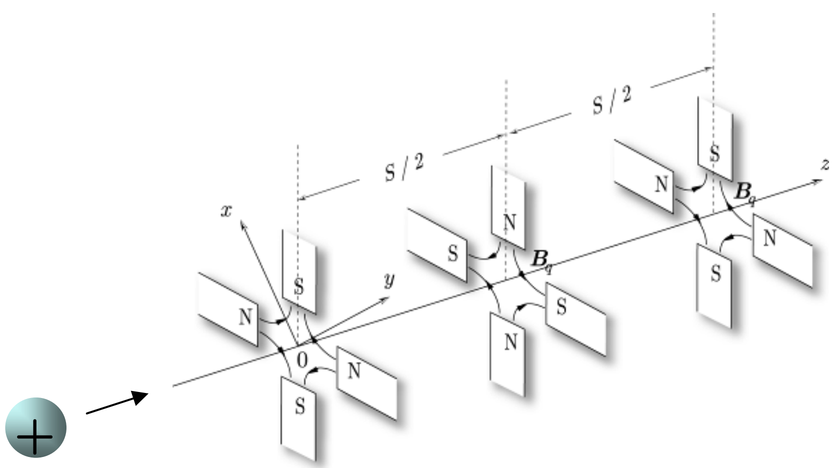


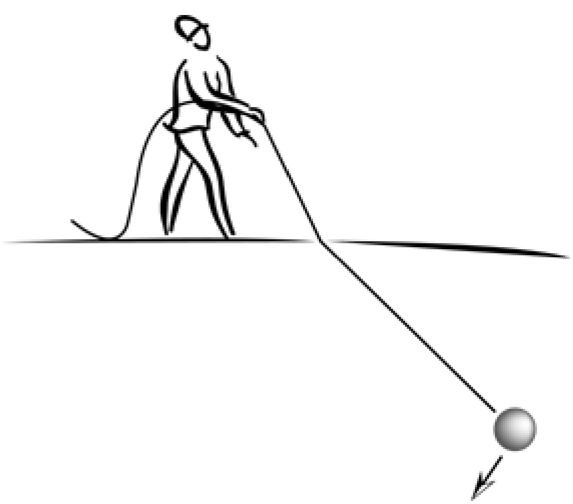
L. Spitzer suggested R. Kulsrud and M. Kruskal to look at a simpler problem first (1950s).



Particles dynamics in accelerators (uncoupled, 1 degree of freedom)

$$x''(s) + \kappa_q(s)x(s) = 0$$





Courant-Snyder theory for uncoupled dynamics

$$q''(s) + \kappa(s) q(s) = 0$$

Courant (1958)

Courant-Snyder invariant

$$A^2 = \frac{q^2}{w^2} + \left(wq' - w'q\right)^2 = const.$$

$$w''(s) + \kappa(s) w(s) = w^{-3}(s)$$

Envelope eq.

Phase advance

$$\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = M(t) \begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix} \qquad \varphi(t) = \int_0^t \frac{dt}{w^2(t)}$$

$$M(t) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\varphi + \alpha_0 \sin\varphi] & \sqrt{\beta\beta_0}\sin\varphi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\varphi + \frac{\alpha_0 - \alpha}{\sqrt{\beta\beta_0}}\cos\varphi & \sqrt{\frac{\beta_0}{\beta}}[\cos\varphi - \alpha\sin\varphi] \end{pmatrix}$$

Courant-Snyder theory is the best parameterization

- Provides the physics concepts of envelope, phase advance, emittance, C-S invariant, KV beam, …

K. Takayama [82,83,92]

Higher dimensions? 2D coupled transverse dynamics?

$$H_c = z A_c z^T, \quad z = (x, y, \dot{x}, \dot{y})$$

$$A_c = \begin{pmatrix} \kappa & R \\ R^T & \frac{I}{2} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}$$

What is $M(t)$?

--- $M(t) \in Sp(4, \mathbb{R})$

10 free parameters

solenoidal, quadrupole,
& skew-quadrupole

$$\kappa = \begin{pmatrix} \frac{\Omega^2}{2} + \frac{\kappa_q}{2} & \kappa_{sq} \\ \kappa_{sq} & \frac{\Omega^2}{2} - \frac{\kappa_q}{2} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -\frac{\Omega}{2} \\ \frac{\Omega}{2} & 0 \end{pmatrix}$$

Many ways [Teng, 71] to parameterize the transfer matrix

Lee Teng

Symplectic rotation form [Edward-Teng, 73]:

$$Z = F^{-1} z$$

$$F = \begin{pmatrix} I\cos\phi & D^{-1}\sin\phi \\ -D\sin\phi & I\cos\phi \end{pmatrix}$$

uncoupled

No apparent physical meaning

No β function

$$M(t) = F \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} F^{-1}$$

uncoupled CS transfer matrix

Have to define beta function from particle trajectories [Ripken, 70], [Wiedemann, 99]

Can we do better? A hint from 1 DOF C-S theory

$$M(t)=\begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\varphi+\alpha_0\sin\varphi] & \sqrt{\beta\beta_0}\sin\varphi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\varphi+\frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\varphi & \sqrt{\frac{\beta_0}{\beta}}[\cos\varphi-\alpha\sin\varphi] \end{pmatrix}$$

$$\beta(t)=w^2(t),\ \alpha(t)=-w\dot{w},\ \varphi(t)=\int_0^t\frac{dt}{\beta(t)}.$$

$$M(t)=\begin{pmatrix} w & 0 \\ \dot{w} & \frac{1}{w} \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

Transfer matrix

Original Courant-Snyder theory

$P \in SO(2)$

$$M(t) = \begin{pmatrix} w & 0 \\ \dot{w} & \dfrac{1}{w} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

scalar

Non-Abelian generalization

$$M(t) = \begin{pmatrix} w^T & 0 \\ w^{-1}\dot{w}w^T & w^{-1} \end{pmatrix} \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \begin{pmatrix} \left(w_0^{-1}\right)^T & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

2×2

$P \in SO(4)$

Envelope equation

Original Courant-Snyder theory

scalar

$$w''(s) + \kappa(s)w(s) = w^{-3}(s)$$

Non-Abelian generalization

$$w''(s) + w(s)\kappa(s) = \left(w^{-1}\right)^T w^{-1}\left(w^{-1}\right)^T$$

2×2

Phase advance rate

Original Courant-Snyder theory

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -w^{-2} \\ w^{-2} & 0 \end{pmatrix} \in so(2)$$

↓

Non-Abelian generalization

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -\left(w^{-1}\right)^T w^{-1} \\ \left(w^{-1}\right)^T w^{-1} & 0 \end{pmatrix} \in so(4)$$

Phase advance

Original Courant-Snyder theory

$$\dot{P} = P\dot{\varphi}$$

$$P = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$$

↓

Non-Abelian generalization

$$\dot{P} = P\dot{\varphi}$$

$$P = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \in SO(4)$$

Courant-Snyder Invariant

Original Courant-Snyder theory

$$I = \frac{q^2}{w^2} + \left(w\dot{q} - \dot{w}q\right)^2 = \left(q, \dot{q}\right)\begin{pmatrix} w^{-1} & -\dot{w} \\ 0 & w \end{pmatrix}\begin{pmatrix} w^{-1} & 0 \\ -\dot{w} & w \end{pmatrix}\begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

↓

Non-Abelian generalization

$$I = \left(x^T, \dot{x}^T\right)\begin{pmatrix} w^{-1} & -\dot{w}^T \\ 0 & w^T \end{pmatrix}\begin{pmatrix} w^{-1T} & 0 \\ -\dot{w} & w \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

How did we do it? General problem

$2n \times 2n$

$$H = z^T A(t) z$$

$$z = (x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n)^T$$

Hamiltonian Eq.

$$z = J \nabla H, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Time-dependent canonical transformation

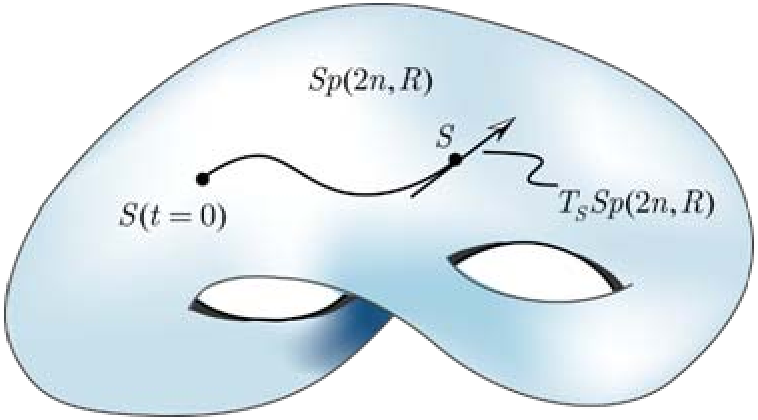
$$\overline{z} = S(t)z$$

$$\overline{H} = \overline{z}^T \overline{A}(t) \overline{z}$$

Target Hamiltonian

$$\dot{S} = 2\left(J\overline{A}S - SJA\right)$$

S can be chosen to be symplectic: $SJS^T = J$



symplectic group

Non-Abelian Courant-Snyder theory for coupled transverse dynamics

Step I: envelope

$$H_c = z^T A_c z,\ z = (x, y, \dot{x}, \dot{y})^T$$

$$A_c = \begin{pmatrix} \kappa & 0 \\ 0 & \frac{I}{2} \end{pmatrix},\ \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}$$

$$\overline{H}_c = \overline{z}^T \overline{A}_c \overline{z},\ \overline{A}_c = \begin{pmatrix} \frac{\beta_I}{2} & 0 \\ 0 & \frac{\beta_I}{2} \end{pmatrix}$$

$$\dot{S} = 2\left(J\overline{A}_c S - SJA_c\right)$$

$$\begin{pmatrix} \dot{S}_1 & \dot{S}_2 \\ \dot{S}_3 & \dot{S}_4 \end{pmatrix} = 2\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\beta_I}{2} & 0 \\ 0 & \frac{\beta_I}{2} \end{pmatrix}\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} - 2\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\kappa}{2} & 0 \\ 0 & \frac{I}{2} \end{pmatrix}$$

$$S_2 = 0$$

$$\ddot{S}_4 = \beta_I^2 S_4 - S_4 \kappa$$

$$w \equiv S_4$$

$$\beta \equiv \beta_I^{-1}$$

$$\ddot{w} = \beta^{-2} w - w\kappa$$

S is symplectic:

$$\beta^{-1} w w^T = I$$

Non-Abelian envelope Eq.

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^T w^{-1} \left(w^{-1}\right)^T$$

2×2

$$\overline{z} = Sz$$

$$S = \begin{pmatrix} \left(w^{-1}\right)^T & 0 \\ -\dot{w} & w \end{pmatrix}$$

2×2

$$S^{-1} = \begin{pmatrix} w^T & 0 \\ w^{-1}\dot{w}w^T & w^{-1} \end{pmatrix}$$

Step II: phase advance

$$\overline{H}_c = \overline{z}^T \overline{A}_c \overline{z}\,,\ \ \overline{A}_c = \begin{pmatrix} \dfrac{w^{-1}}{2} & 0 \\ 0 & \dfrac{w^{-1}}{2} \end{pmatrix}$$

$$\overline{\overline{z}} = P\overline{z}$$

$$\overline{\overline{H}}_c = 0$$

$$\dot{P} = -2PJ\overline{A}_c$$

$$P = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix} \in SO(4)$$

$$\dot{P} = P\dot{\varphi}$$

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -\left(w^{-1}\right)^T w^{-1} \\ \left(w^{-1}\right)^T w^{-1} & 0 \end{pmatrix} \in so(4)$$

Application: Strongly coupled system
Stability completely determined by phase advance

Suggested by K. Takayama

Theorem 1:

Unstable $\Leftrightarrow P_c(t)$ has an eigenvalue with $|\lambda| \neq 1$

one turn map

Theorem 2:

Stable $\Leftarrow J[P_c^T(t) - P_c(t)]$ is positive (negative)-definite

Application: Weakly coupled system
Stability determined by uncoupled phase advance

Theorem 1:

Unstable $\Leftrightarrow P_c(t)$ has an eigenvalue with $|\lambda| \neq 1$

↓

Unstable $\Rightarrow$ uncoupled tune $\cos\phi_x = \cos\phi_y$

↕

uncoupled tune $\nu_x \pm \nu_y = n$

(sum and difference resonace)

Application: Weakly coupled system
Stability determined by uncoupled phase advance

Theorem 2:

Stable $\Leftarrow J[P_c^T(t) - P_c(t)]$ is positive (negative)-definite

$\downarrow$

$$\text{stable} \Leftarrow \sin\phi_x = \sin\phi_y$$

$\downarrow$

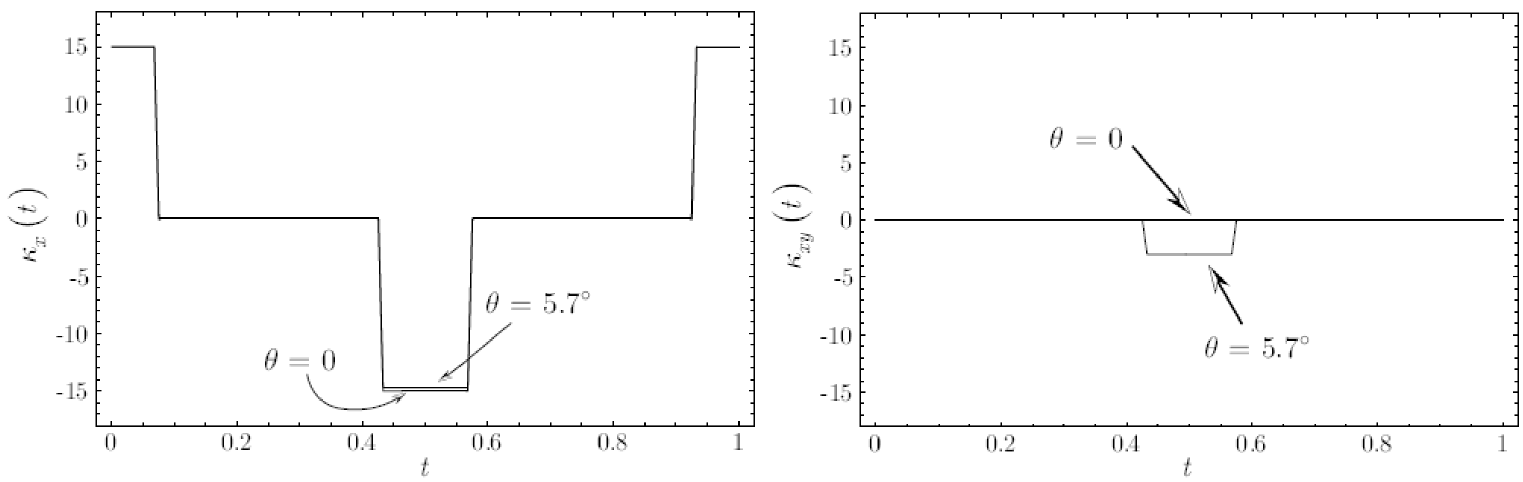
$$\text{unstable} \Rightarrow \nu_x + \nu_y = n \ \text{(sum resonace)}$$

Numerical example – mis-aligned FODO lattice

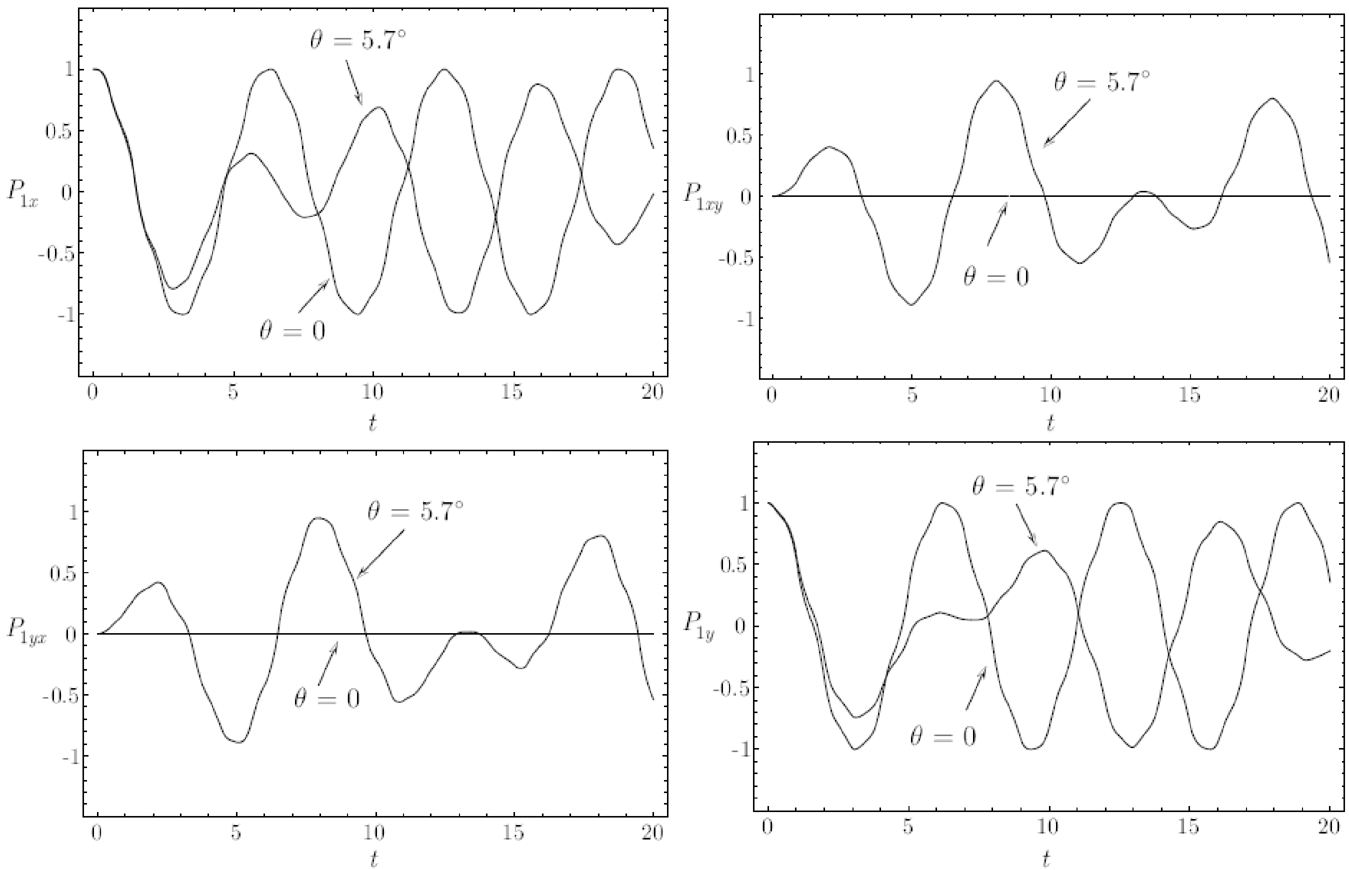
mis-alignment angle

$$\kappa = \kappa_q \begin{pmatrix} \cos[2\theta(s)] & \sin[2\theta(s)] \\ \sin[2\theta(s)] & -\cos[2\theta(s)] \end{pmatrix}$$

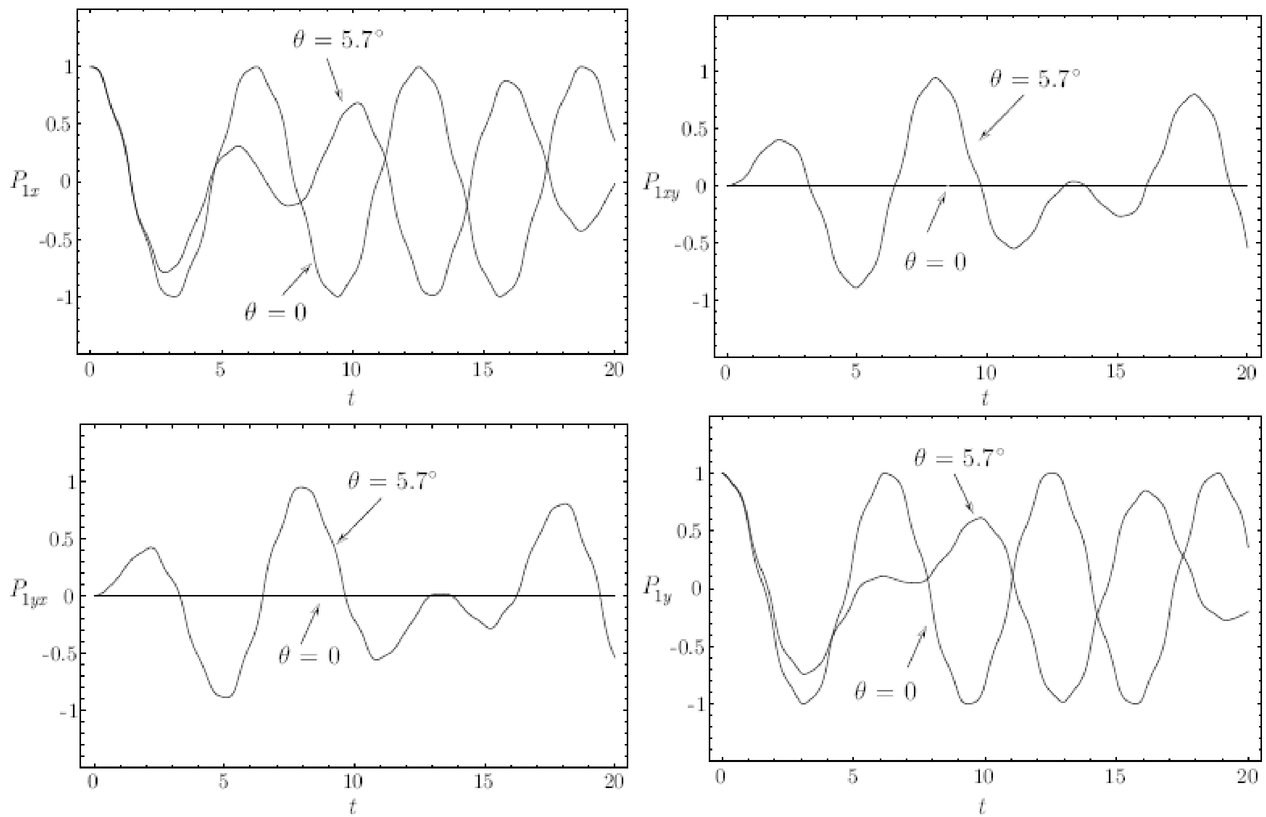
[Barnard, 96]



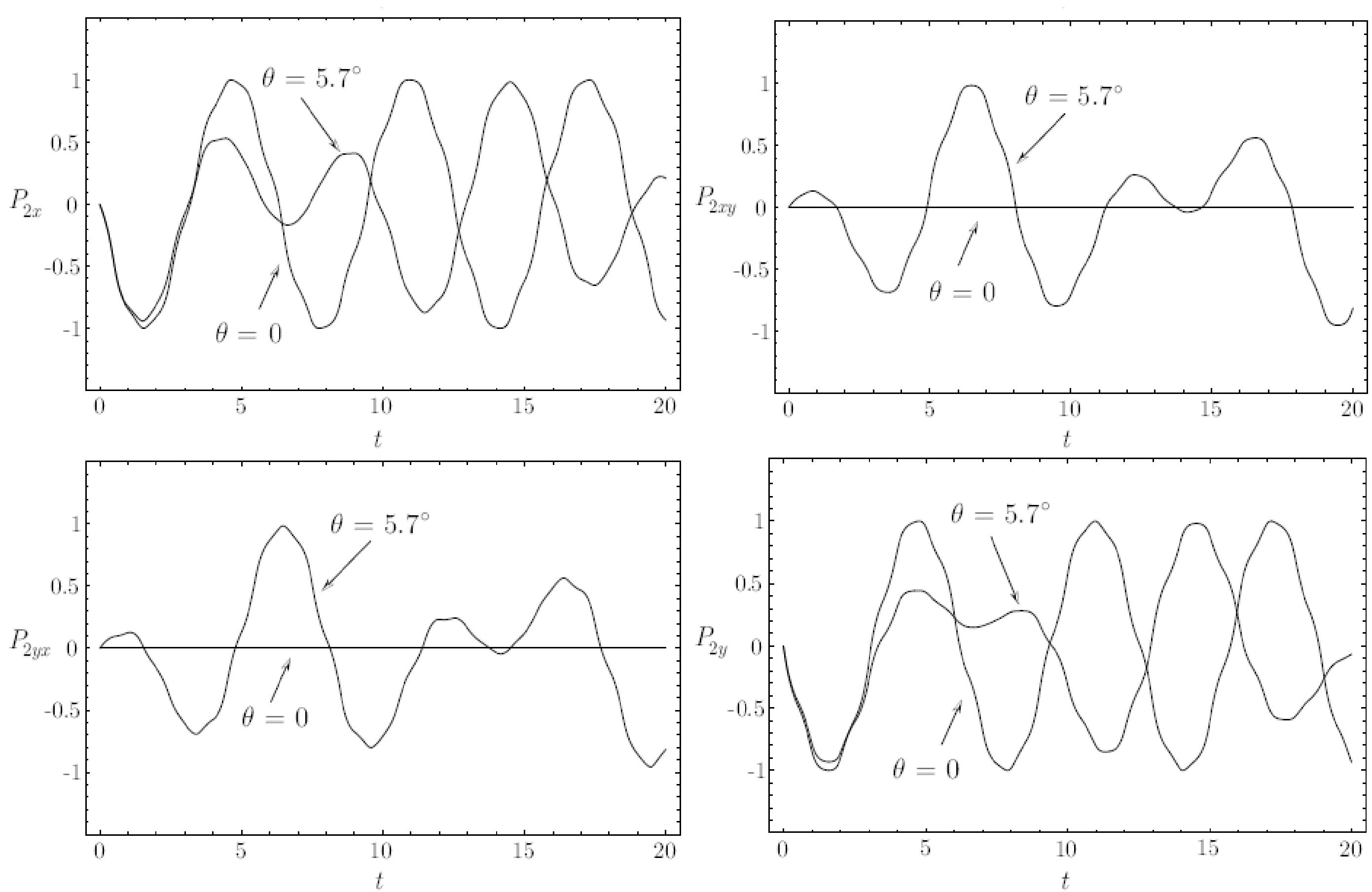
Envelope matrix w



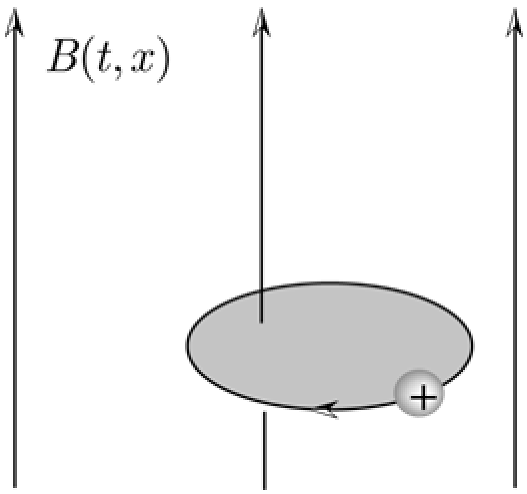
Rotation matrix P_1



Rotation matrix P_2



Other appliations – exact invariant of magnetic moment



Theorem 1. For an arbitrary function $\kappa(t)$ and w_1, w_2 satisfying

$$\ddot{w}_1 + \kappa w_1 = \frac{\varepsilon_1}{w_1^3},$$

$$\ddot{w}_2 + \kappa w_2 = \frac{\varepsilon_2}{w_2^3},$$

where ε_1 and ε_2 are real constants, the quantity

$$I = \varepsilon_1 \left(\frac{w_2}{w_1} \right)^2 + \varepsilon_2 \left(\frac{w_1}{w_2} \right)^2 + \left(w_2 \dot{w}_1 - \dot{w}_2 w_1 \right)^2$$

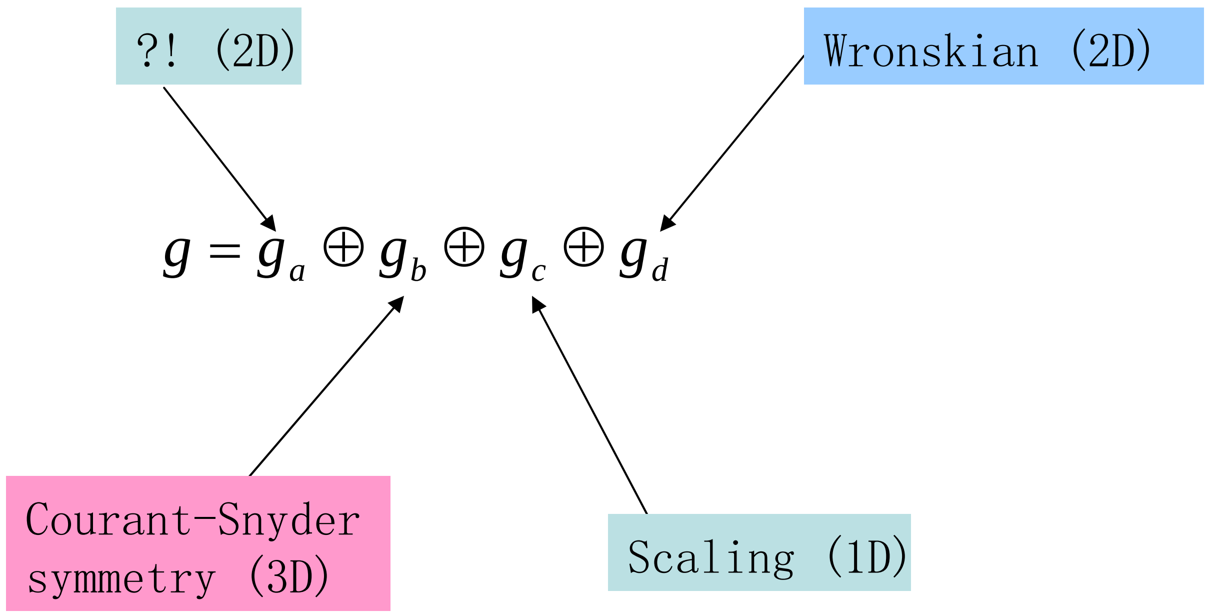
is an invariant.

[K. Takayama, 92]
[H. Qin and R. C. Davidson, 06]

Other applications

- Globally strongly coupled beams. (many-fold Möbius accelerator).
- 4D emittance [Barnard, 96].
- 4D KV beams.
- …

Other applications – symmetry group of 1D time-dependent oscillator



[H. Qin and R. C. Davidson, 06]